** S O L U T I O N S * * *

University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

MATA32 - Midterm Test - Calculus for Management I

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Date: October 22, 2011
Time: 9:00 am
Duration: 110 minutes

Clearly indicate the following information:

Last Name (Print): ______________________

Given Name(s)(Print): ______________________

Student Number: ______________________

Signature: * S O L U T I O N S *

Tutorial Number (e.g. TUT0032): ______________________

Carefully circle the name of your Teaching Assistant:

Natalia CHERNEGA	Yinzheng (Jerry) GU	Xin (Viki) QI
Jaehyun CHO	Winnie LAM	Mitsuru WILSON
Chris CHOW	Yik Chau (Kry) LUI	Tianhui (Alan) WU
Fazle CHOWDHURY	Pourya MEMARPANAHI	Wanying ZHU

Read these instructions:

1. This test has ~~11~~ 10 (blank intentionally omitted) numbered pages. It is your responsibility to ensure that at the beginning of the test, all of these pages are included.

2. Put your letter answers to Part A (Multiple Choice Questions) in the answer boxes at the top of page 2. Put your solutions and/or rough work to Part B (Full Solution Questions) in the answer spaces provided beneath each question. If you need extra space, use the back of a page ~~or the last page.~~ Clearly indicate the location of your continuing work.

3. You may use **one** standard hand-held calculator (graphing capability is permitted). All other electronic devices, extra paper, notes, and textbooks are forbidden at your workspace (either visibly or in any sort of carrying case or by accident).

4. You are strongly encouraged to **write your test in pen or other ink**. Tests written in pencil will be denied any remarking or revision privilege.

* S O L U T I O N S *

Print letters for Part A (Multiple Choice Questions) in these boxes.

1	2	3	4	5	6	7
C	A	D	C	A	E	D

Do not write anything in the boxes below.

ALL cover info present ⇒ 3 points
ANY cover info missing ⇒ 0 points

Info.	Part A
3	21

Part B

1	2	3	4	5	6
11	10	10	14	14	17

Total
100

The following formulas may be helpful:

$S = P(1+r)^n$ $\quad S = Pe^{rt}$ $\quad$ revenue = price × quantity

For ordinary annuity: $S = R\left[\frac{(1+r)^n - 1}{r}\right]$ and $A = R\left[\frac{1-(1+r)^{-n}}{r}\right]$

For annuity due: $S = R\left[\frac{(1+r)^{n+1} - 1}{r}\right] - R$ and $A = R + R\left[\frac{1-(1+r)^{-n+1}}{r}\right]$

Part A (Multiple Choice Questions) For each of the following **clearly print the letter of the answer you think is most correct in the boxes at the top of page 2.** Each right answer earns 3 points and no answer/wrong answers earn 0 points. Justification is neither required nor rewarded, but a small workspace is provided for your calculations.

1. What annual percentage rate of interest, compounded semi-annually, is equivalent to 3.78% compounded continuously ?

   (A) 4.161% (B) 3.908% (C) 3.816% (D) 3.798%
   (E) a percentage that is not in (A) - (D)

$$\left(1+\frac{r}{2}\right)^2 = e^{.0378}$$

$$\therefore \quad r = 2\left(e^{(.0378)(.5)} - 1\right) \approx .03816$$

As a %, ≈ 3.816

2. An apartment costs $1,300 per month to rent and is payable at the start of each month. Interest is 2.4% APR compounding monthly. What would be the total amount of one year's rent (rounded up to the nearest dollar) if you were to pay all of this at the beginning of the year ?

   (A) $15,430 (B) $15,440 (C) $15,502 (D) $15,510
   (E) an amount not in (A) - (D)

$$A = 1,300 + 1,300\left[\frac{1-(1.002)^{-11}}{.002}\right] \approx 15,429.88$$

$$r = \frac{.024}{12} = .002$$

3. The slope of the curve $y = \dfrac{\sqrt{x}(81-x^2)}{x}$ at the point $(9,0)$ is

   (A) 0 (B) −1.5 (C) −3 (D) −6 (E) a number not in (A) - (D)
   (F) undefined

$$y = \frac{81-x^2}{\sqrt{x}} = 81x^{-1/2} - x^{3/2}$$

$$\therefore \quad y' = -\frac{81}{2}x^{-3/2} - \frac{3}{2}x^{1/2} \qquad y'(9) = \left(-\frac{81}{2}\right)\left(\frac{1}{9^{3/2}}\right) - \frac{9}{2}$$

$$= -\frac{3}{2} - \frac{9}{2} = -6$$

4. If $f(x) = \dfrac{2(x^2+6x-16)}{x^3-2x^2} + ln\left(e^3 - x + 2\right)$ then $\lim_{x\to 2} f(x)$ equals

(A) -2 (B) 5 (C) 8 (D) 0 (E) a number not (A) - (D)

Factor: $\dfrac{2(x-2)(x+8)}{x^2(x-2)} \longrightarrow \dfrac{2(10)}{4} = 5$ (by limit props)

$\ln(e^3-x+2)$ cts @ 2 $\longrightarrow \ln(e^3) = 3$ $5+3=8$

5. A manufacturer's average cost function is given by $\bar{c} = 56 + \dfrac{48}{q^2}$ where $q > 0$ is quantity. The marginal cost when $q = 4$ is

(A) 53 (B) 59 (C) 227 (D) -0.75 (E) -1.5 (F) a number not in (A) - (E)

cost $c = \bar{c}\cdot q = 56q + \dfrac{48}{q}$

$c'(q) = 56 - \dfrac{48}{q^2}$ $c'(4) = 56 - \dfrac{48}{16} = 53$

6. If $a > 0$ is a constant and $y = \dfrac{a-x}{a+x} + ln(x)$ then the value of $\left.\dfrac{dy}{dx}\right|_{x=a}$ is

(A) $\dfrac{-1}{2a}$ (B) 0 (C) $\dfrac{-2}{a}$ (D) $\dfrac{2}{a}$ (E) $\dfrac{1}{2a}$ (F) a number not in (A) - (E)

$y' = \dfrac{(-1)(a+x)-(a-x)}{(a+x)^2} + \dfrac{1}{x}$

$\therefore\ y'(a) = \dfrac{-2a}{(2a)^2} + \dfrac{1}{a} = -\dfrac{1}{2a} + \dfrac{2}{2a} = \dfrac{1}{2a}$

7. The least number of months it takes an investment to increase by 28% at a periodic rate of 0.5% compounding at the end of every other month is

(A) 12 (B) 50 (C) 99 (D) 100 (E) 198 (F) a number not in (A) - (E)

Solve $1.28 = (1.005)^{6t}$ $t =$ years

$\ln(1.28) = 6t\ln(1.005)$

Get $t = \dfrac{\ln(1.28)}{6\ln(1.005)} \approx 8.24922$ years ≈ 98.99064 months

Make sure that your answers are printed in the letter boxes at the top of page 2

However: compounding occurs only every other month (so 99 is invalid).

$\therefore$ answer is 100 months.

Part B (Full Solution Questions) Write clear and neat solutions in the answer spaces provided. Show all of your work. Full points are awarded only for solutions that are correct, complete, and sufficiently display relevant concepts from MATA32.

1. Evaluate the following limits. Use the ∞ or $-\infty$ symbol where appropriate.

(a) $\lim_{x\to\infty}\left(\frac{4x^2-1}{(2x+3)^2} + 6\sqrt{\frac{x+3}{4x+9}} + e^{1/x}\right)$ $= 1+3+1 = 5$ [4 points]

$\frac{4x^2-1}{4x^2+12x+9} \longrightarrow \frac{4}{4} = 1$ as $x \to \infty$

Answer:

$1+3+1=5$

by limit props

$6\sqrt{\frac{x+3}{4x+9}} \longrightarrow 6\sqrt{\frac{1}{4}} = \frac{6}{2}$ as $x\to\infty$

$\frac{1}{x} \to 0$ as $x \to \infty$ so $e^{1/x} \to e^{0} = 1$

(b) $\lim_{x\to 1^-}\left(\frac{1-x^3}{1-\sqrt{x}}\right)$ $= \lim_{x\to 1^-} \frac{(1-x)(1+x+x^2)}{1-\sqrt{x}}$ [4 points]

$= \lim_{x\to 1^-} \frac{(1-\sqrt{x})(1+\sqrt{x})(1+x+x^2)}{(1-\sqrt{x})}$ (Cancel $1-\sqrt{x}$)

$= \lim_{x\to 1^-}\left[(1+\sqrt{x})(1+x+x^2)\right]$ (Continuity)

$= (1+1)(1+1+1) = 6$

(c) $\lim_{u\to 0}\left(\frac{u+1}{u^3-u^2}\right)$ $= \lim_{u\to 0}\frac{u+1}{u^2(u-1)} = -\infty$ [3 points]

Reasoning: As $u\to 0$, $u+1 \to 1$ and $u^2 \to 0$

Also, $u-1 \to -1$

$\therefore$ (top) $\to 1$ (bottom) $\to 0^-$ (tends to 0 from left)

$\therefore \frac{\text{top}}{\text{bottom}} \to -\infty$

2. A debt of \$4,000 due at the end of 3 years from now and \$9,000 due at the end of 5 years after that is to be repaid by three payments:

(1) a first payment now;

3 + 5 = 8 years total

(2) a second payment at the end of 2 years from now of an amount \$1,000 more than the first payment;

(3) a third payment at the end of 4 years from now of an amount \$2,000 more than the second payment.

Interest is 3% APR compounding semi-annually. Find the amount of each payment rounded up to the nearest dollar. A complete solution includes a labeled money-time diagram and a clear equation of value. [10 points]

Let all money be in units of \$1,000.

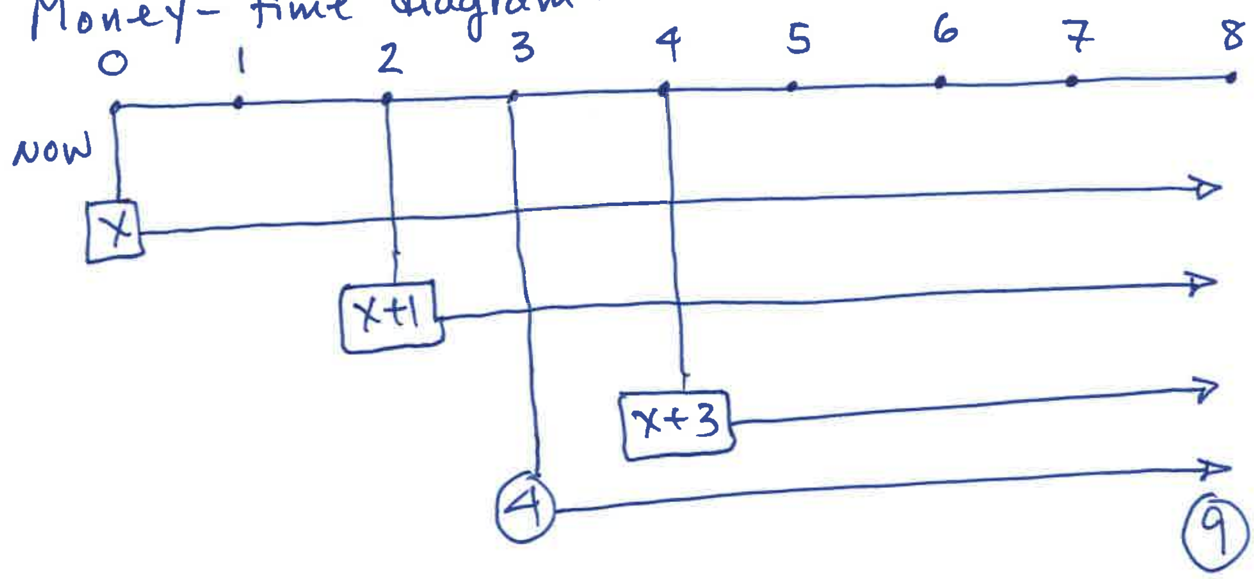
debt

$\boxed{\text{pay}}$

$r = \frac{.03}{2} = .015$

X = Amount of 1ST payment

Money-time diagram:



Equation of Value:

$$x(1.015)^{16} + (x+1)(1.015)^{12} + (x+3)(1.015)^{8} = 4(1.015)^{10} + 9$$

We get

$$3.591096257\,x = 9.067067369$$

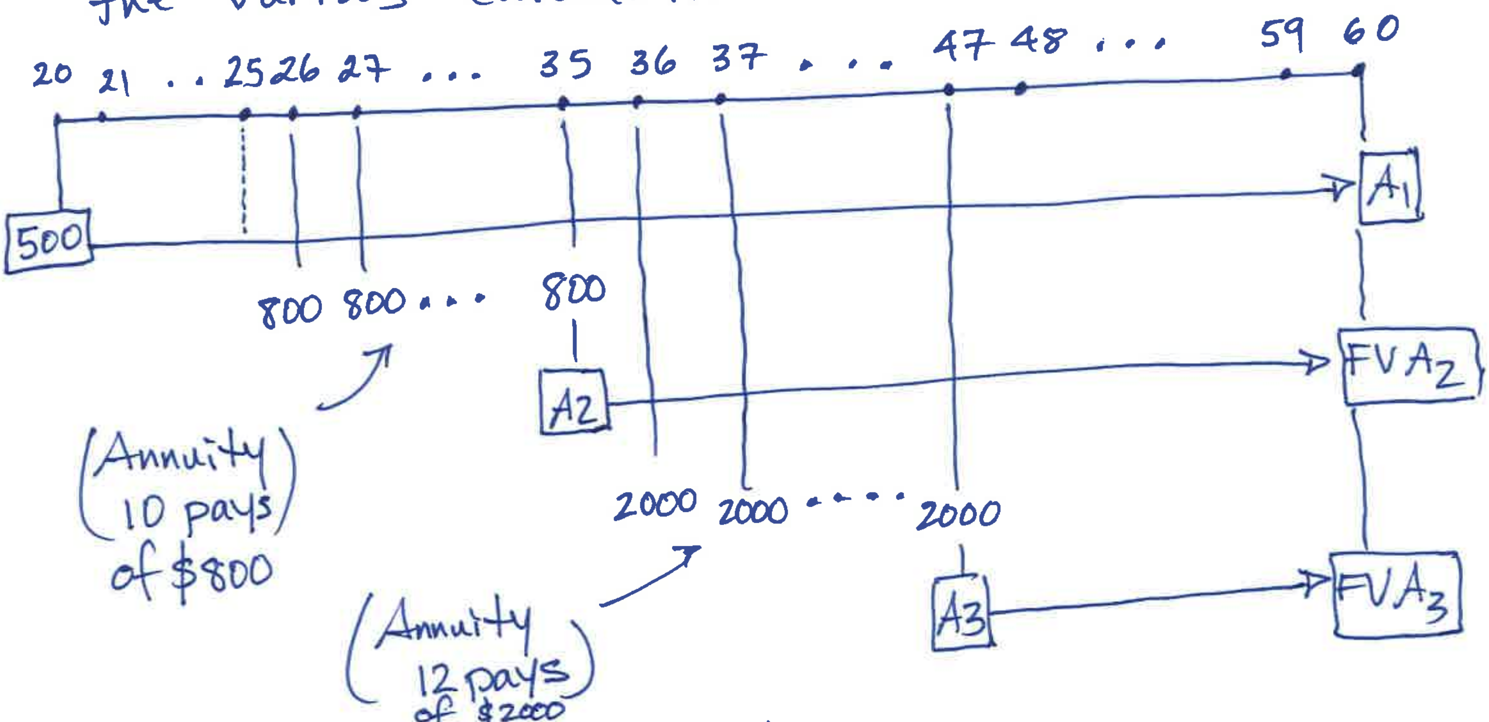
$$\therefore x = 2.524874501$$

Payments (with rounding):

1ST = 2,525
2nd = 3,525
3rd = 5,525

3. In all of the following investment situation, the account has interest of 2% compounding annually. On your 20^{th} birthday you deposit \$500 into an empty account. Starting on your 26^{th} birthday and ending on your 35^{th} birthday, you deposit \$800 into the same account on each birthday. Then beginning with your 36^{th} birthday and ending on your 47^{th} birthday, you deposit \$2,000 into the same account on each birthday. No further deposits are made after your 47^{th} birthday. Calculate how much you will have in the account (rounded up to the nearest dollar) on your 60^{th} birthday. [10 points]

Cash-time diagram that helps to organize the various calculations.



TOTAL $= A_1 + FVA_2 + FVA_3$

$$A_1 = 500(1.02)^{40} \approx 1{,}104.0198$$

$$FVA_2 = (A_2)(1.02)^{25} = 800\left[\frac{(1.02)^{10}-1}{.02}\right](1.02)^{25}$$

$$\approx 14{,}371.3423$$

$$FVA_3 = (A_3)(1.02)^{13} = 2{,}000\left[\frac{(1.02)^{12}-1}{.02}\right](1.02)^{13}$$

$$\approx 34{,}699.9364$$

$\therefore$ TOTAL $= 50{,}176$ with rounding

4. (a) If $x, a > 0$ and $w = g(x) = x^2 e^{ax}$, show that $\dfrac{dw}{dx} = w\left[\dfrac{2}{x} + a\right]$ [3 points]

$$\frac{dw}{dx} = 2xe^{ax} + x^2 e^{ax} a$$

$$= \frac{2x^2 e^{ax}}{x} + a x^2 e^{ax} = x^2 e^{ax}\left[\frac{2}{x} + a\right] = w\left[\frac{2}{x} + a\right]$$

(b) If $y = 4u^3 - 5u^2 + 7$ and $u = x^2 + 2\sqrt{x}$, find the exact value of $y'(1)$ where y' means differentiation with respect to x. [5 points]

$u(1) = 3$

$$y'(1) = \left.\frac{dy}{dx}\right|_{x=1} = \left.\frac{dy}{du}\right|_{u=3} \cdot \left.\frac{du}{dx}\right|_{x=1} \quad \text{(by Chain rule)}$$

$$= \left.(12u^2 - 10u)\right|_{u=3} \cdot \left.\left(2x + \frac{1}{\sqrt{x}}\right)\right|_{x=1}$$

$$= (108 - 30)(2 + 1) = (78)(3) = 234$$

(c) A manufacturer's price function is given by $p = \dfrac{1000}{q+5}$ where p is in dollars per unit and q is quantity (i.e. numbers of units). Find the marginal revenue when $q = 45$. [6 points]

Revenue $r = pq = \dfrac{1000q}{q+5}$

Marginal revenue $= r'(q) = \dfrac{d}{dq}\left(\dfrac{1000q}{q+5}\right)$

$$= \frac{1000(q+5) - 1000q}{(q+5)^2}$$

$$r'(45) = \frac{1000(50) - 1000(45)}{(50)^2} = \frac{5000}{2500} = 2$$

5. In all of this question, let $y = f(x) = \dfrac{-6}{1-x^2}$ Undefined when $x = \pm 1$

(a) State the domain of f using interval notation. [3 points]

$(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

(b) Differentiate f and write your answer as a rational function. [4 points]

$$f'(x) = 6(1-x^2)^{-2}(-2x)$$

$$= \frac{-12x}{(1-x^2)^2}$$

(c) Use the definition of derivative to calculate $f'(x)$ [7 points]

$$f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{\dfrac{-6}{1-(x+h)^2} - \dfrac{-6}{1-x^2}}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{-6(1-x^2) + 6[1-(x+h)^2]}{h[1-(x+h)^2][1-x^2]} \right]$$

$$= \lim_{h \to 0} \left[\frac{-6+6x^2+6-6x^2-12xh-6h^2}{h[1-(x+h)^2][1-x^2]} \right]$$

$$= \lim_{h \to 0} \frac{-12x-6h}{[1-(x+h)^2][1-x^2]} = \frac{-12x}{(1-x^2)^2}$$

cts. = continuous

6. In all of this question c is a constant and

$$h(x) = \begin{cases} x^2 + 4x + 3 & if \quad x \le 1 \\ c^2x + c & if \quad x > 1 \end{cases}$$

(a) Explain briefly why h is continuous on the intervals $(-\infty, 1)$ and $(1, \infty)$. [3 points]

For $x \in (-\infty, 1)$, $h(x) = x^2+4x+3$ ... polynomial $\rightarrow$ cts.

For $x \in (1, \infty)$, $h(x) = c^2x + c$ ... polynomial $\rightarrow$ cts.

(note $h(x)$ is a polynomial for $x > 1$ and any value of c)

(b) Find the value(s) of c that makes $\underbrace{h \text{ continuous at } x = 1}_{(*)}$. [5 points]

(*) if and only if $\lim_{x \to 1} h(x) = h(1)$

$h(1) = 8$

Solve $c^2 + c = 8$

Get

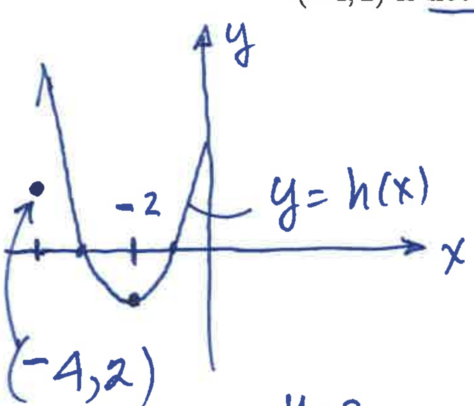
$c = \dfrac{-1 \pm \sqrt{33}}{2}$

$\lim_{x \to 1^-} h(x) = \lim_{x \to 1^-} x^2 + 4x + 3 = 8$

$\lim_{x \to 1^+} h(x) = \lim_{x \to 1^+} c^2x + c = c^2 + c$

Set =

(c) When $x < 1$, there are two tangent lines to the curve $y = h(x)$ that pass through the point $(-4, 2)$. Find the equation of the one having largest slope. Note that the point $(-4, 2)$ is not on the curve. [9 points]



For $x \le 1$, $h(x) = x^2 + 4x + 4 - 1$

$= (x+2)^2 - 1$

Find the slope in "two ways"; equate; solve. Let (x, y) be a point of tangency

$\dfrac{y-2}{x+4} = h'(x) = 2x+4$

$\rightarrow x^2 + 4x + 1 = (2x+4)(x+4)$

$\therefore x^2 + 8x + 15 = 0$

$(x+3)(x+5) = 0$

$\therefore x = -3, \ x = -5$

$h'(-3) = -2 \leftarrow$ LARGER

$h'(-5) = -6$

$\therefore$ Largest slope $= -2$

$y = h(-3) = 0$

Point $(x, y) = (-3, 0)$

Equation:

$y - 0 = -2(x+3)$

Final Answer

$y = -2x - 6$