*** SOLUTIONS ***

University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

MATA32 - Midterm Test - Calculus for Management I

Examiner: R. Grinnell

Date: February 18, 2012
Duration: 120 minutes
Time: 9:00 am

Clearly indicate the following information:

Last Name (Print): ______________________

Given Name(s)(Print): ______________________

Student Number: ______________________

Signature: SOLUTIONS and STATISTICS

Tutorial Number (e.g. TUT0032): ______________________

Carefully circle the name of your Teaching Assistant:

Fazle CHOWDHURY   Yik Chau (Kry) LUI   Kevin YAN

Read these instructions:

1. This test has 10 numbered pages. It is your responsibility to ensure that at the beginning of the test, all of these pages are included.
2. Put your letter answers to Part A (Multiple Choice Questions) in the answer boxes at the top of page 2. Put your solutions and/or rough work to Part B (Full Solution Questions) in the answer spaces provided beneath each question. If you need extra space, use the back of a page or the last page. Clearly indicate the location of your continuing work.
3. You may use **one** standard hand-held calculator. All other electronic devices, extra paper, notes, and textbooks are forbidden at your workspace (either visibly or in any sort of carrying case or by accident).
4. You are strongly encouraged to write your test in pen or other ink. Tests written in pencil will be denied any remarking or revision privilege.

*** SOLUTIONS ***

Print letters for Part A (Multiple Choice Questions) in these boxes.

1	2	3	4	5	6	7
D	D	C	B	B	E	A

Do not write anything in the boxes below.

Info.	**Part A**
2	21

Part B

1	2	3	4	5
15	16	16	17	13

Total
100

The following formulas may be helpful:

$$S = P(1+r)^n \qquad S = Pe^{rt} \qquad \text{revenue} = \text{price} \times \text{quantity}$$

For ordinary annuity: $S = R\left[\frac{(1+r)^n - 1}{r}\right]$ and $A = R\left[\frac{1-(1+r)^{-n}}{r}\right]$

For annuity due: $S = R\left[\frac{(1+r)^{n+1} - 1}{r}\right] - R$ and $A = R + R\left[\frac{1-(1+r)^{-n+1}}{r}\right]$

Part A (Multiple Choice Questions) For each of the following **clearly print the letter of the answer you think is most correct in the boxes at the top of page 2.** Each right answer earns 3 points and no answer/wrong answers earn 0 points. Justification is neither required nor rewarded, but a small workspace is provided for your calculations.

1. If $f(x) = \dfrac{7x^3 + 20x}{5\sqrt{x}}$ then $f'(4)$ equals

(A) 52.8 (B) 42.2 (C) 36 (D) 29 (E) a number not in (A) - (D)

$f(x) = \frac{7}{5}x^{5/2} + 4x^{1/2}$

$\therefore f'(x) = \frac{7}{2}x^{3/2} + 2x^{-1/2}$

$f'(4) = \frac{7}{2}(4)^{3/2} + 2(4)^{-1/2}$

$= \frac{7}{2}(8) + \frac{2}{2} = 29$

2. The present value of $2,460 due in 11.5 years at 1.8% APR compounding continuously is (rounded down to the nearest dollar)

(A) $3,025 (B) $2,157 (C) $2,032 (D) $2,000 (E) none of (A) - (D)

$PV = 2{,}460e^{(-11.5)(0.018)}$

$\approx 2{,}000.028$

3. If $R(x) = \dfrac{x^3 - x}{x^3 - x^2 + x - 1} + ln\big((2x-1)e\big)$ then $\lim\limits_{x\to 1} R(x)$ equals

(A) 0 (B) 1 (C) 2 (D) $1 + e$ (E) none of (A) - (D)

$\dfrac{1^3 - 1}{1^3 - 1^2 + 1 - 1} = \dfrac{0}{0} \rightarrow$ factor/simplify

$\dfrac{x^3 - x}{x^3 - x^2 + x - 1} = \dfrac{x(x-1)(x+1)}{x^2(x-1) + 1(x-1)} = \dfrac{x\cancel{(x-1)}(x+1)}{(x^2+1)\cancel{(x-1)}} = \dfrac{x(x+1)}{x^2+1}$ (when $x \neq 1$)

$\therefore \lim\limits_{x\to 1} R(x) = \left[\lim\limits_{x\to 1} \dfrac{x(x+1)}{x^2+1}\right] + \ln\big((2-1)e\big) = \frac{2}{2} + 1 = 2$

4. The future value of an annuity due consisting of twenty-five \$300 payments made at the beginning of each month at a rate of 2.4% APR compounding monthly is (rounded down to the nearest dollar)

(A) \$7,998 (B) \$7,698 (C) \$7,682 (D) \$7,382 (E) not any of (A) - (D)

$$S = 300\left[\frac{(1.002)^{26}-1}{.002}\right] - 300$$

$$r = \frac{2.4}{100} \times \frac{1}{12} = .002$$

$$\approx 7,698.156$$

5. The value of the constant c that makes $\lim_{x\to-\infty}\left(\frac{cx^2+x}{4x^2+1}+\frac{\sqrt{x^2+9}}{x}\right)=2$ is

(A) 24 (B) 12 (C) 4 (D) -4 (E) none of (A) - (D)

Evalute the limit in terms of c

$$2 = \lim_{x\to-\infty}\left(\frac{cx^2+x}{4x^2+1}+\frac{\sqrt{x^2+9}}{x}\right)$$

$$= \frac{c}{4} - 1 \Rightarrow \frac{c}{4} = 3 \quad \therefore c = 12$$

6. If the average cost to produce x units of a product is $a(x) = \frac{600}{x+6}$ then the marginal cost to produce 9 units is

(A) about -2.67 (B) 40 (C) about 30.77 (D) 64 (E) 16

C = cost

$$\therefore C(x) = x \cdot a(x) = \frac{600x}{x+6}$$

$$C'(x) = \frac{600(x+6) - 600x}{(x+6)^2} = \frac{3600}{(x+6)^2}$$

$$\therefore C'(9) = \frac{3600}{225} = 16$$

7. The least whole number of months it takes an investment to increase by 10% at 2.25% APR compounding five times annually is

(A) 53 (B) 52 (C) 49 (D) 23 (E) 22 (F) not in (A) - (E)

Let n = # of interest compoundings

$$r = \frac{2.25}{100} \times \frac{1}{5} = .0045$$

Solve $1.1 = (1.0045)^n$

$\Rightarrow n \simeq 21.227659$ compounds

$\Rightarrow$ (22)(2.4) months

(Round-up # of compounds)

Make sure that your answers are printed in the letter boxes at the top of page 2

$(22)(2.4) = 52.8$ months $\rightarrow$ Round-up to 53 whole # of months.

Part B (Full Solution Questions) Write clear and neat solutions in the answer spaces provided. Show all of your work. Full points are awarded only for solutions that are correct, complete, and sufficiently display relevant concepts from MATA32.

1. In all of this question let $y = f(x) = \dfrac{12x}{1+x}$

(a) Find the equation of the tangent line to the curve $y = f(x)$ at the point on the curve where $y = 6$. Give your answer in slope-intercept form. [8 points]

When $y=6$, $6 = \dfrac{12x}{1+x} \Rightarrow 6+6x = 12x$

$\therefore x = 1$ so the point is $(1,6)$

$$f'(x) = \frac{12(1+x) - 12x}{(1+x)^2}$$

$$= \frac{12}{(1+x)^2}$$

$$f'(1) = \frac{12}{4} = 3 = \text{slope.}$$

$$y - 6 = 3(x-1)$$

$$y = 3x - 3 + 6 = 3x + 3$$

$\boxed{\text{Answer: } y = 3x+3}$

(b) Use the definition of derivative (i.e. "first principles") to find $f'(x)$ [7 points]

$$f'(x) = \lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\frac{12(x+h)}{1+x+h} - \frac{12x}{1+x}}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{12(x+h)(1+x) - 12x(1+x+h)}{h(1+x+h)(1+x)}\right]$$

$$= \lim_{h\to 0}\left[\frac{\cancel{12x} + \cancel{12x^2} + 12h + \cancel{12xh} - \cancel{12x} - \cancel{12x^2} - \cancel{12xh}}{h(1+x+h)(1+x)}\right]$$

$$= \lim_{h\to 0}\left[\frac{12}{(1+x+h)(1+x)}\right] = \boxed{\frac{12}{(1+x)^2}}$$

Assume all money is in $1,000

2. (a) A total debt of $5,000 due 5 years from now and $5,000 due 8 years from now is to be repaid with a $1,000 payment now and three more payments as follows:

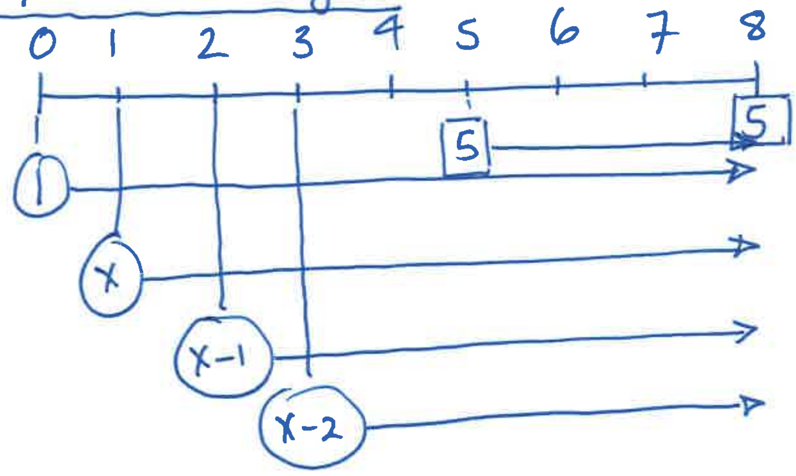
(i) a payment at the end of year 1

(ii) a payment at the end of year 2 of an amount $1,000 less than that in (i)

(iii) a payment at the end of year 3 of an amount of $1,000 less than that in (ii)

Interest is 2.0% APR compounding quarterly. Find the amount of the payments in (i), (ii), and (iii). Round your final answers up to the nearest dollar. A money-time diagram and a well-labeled equation of value is required for full points. [11 points]

Money-time diagram:



$$r = \frac{2}{100} \times \frac{1}{4} = .005$$

Calibrate to end of 8 years.

x = amount of payment @ end of Year 1

Equation of value:

$$(1.005)^{32} + x(1.005)^{28} + (x-1)(1.005)^{24} + (x-2)(1.005)^{20}$$
$$= 5(1.005)^{12} + 5$$

Careful simplification gives the equation

$$3.381927963x = 12.47229687$$
$$\therefore x = 3.687925055$$

Answer:
1st $\approx \$3,688$
2nd $\approx \$2,688$
3rd $\approx \$1,688$

(b) Suppose you buy a computer by paying $200 now and then make $40 payments at the end of every month for a year. Interest is 3.6% APR compounding monthly. Find the "cash price" (i.e. the price of the computer if it were paid for now, in full). Round your answer up to the nearest dollar. [5 points]

$$\text{Cash Price} = 200 + 40\left[\frac{1-(1.003)^{-12}}{0.003}\right]$$

$$\approx 670.7696$$

Answer: $671

3. In all of this question a price function for some product is given by $p = \dfrac{2250}{4q+8}$ where p is in dollars/unit and $q > 0$ is quantity.

(a) Find the marginal revenue when $q = 13$ [7 points]

Revenue $r = pq = \dfrac{2250q}{4q+8}$

Marginal revenue $= \dfrac{dr}{dq} = \dfrac{2250(4q+8) - 2250(4)}{(4q+8)^2}$

$\left.\dfrac{dr}{dq}\right|_{q=13} = \dfrac{2250(60) - 2250(4)}{(60)^2} = \dfrac{18000}{3600} = 5$

Answer: $\boxed{5}$

(b) Let $\mathcal{L} = \lim_{q\to\infty} r$ where r is the revenue function as in (a). Find $\mathcal{L}$ and show that $r < \mathcal{L}$ for all $q > 0$. [5 points]

$\mathcal{L} = \lim_{q\to\infty} r = \lim_{q\to\infty} \dfrac{2250q}{4q+8} = \dfrac{2250}{4} = \boxed{562.5}$

$r = \dfrac{2250q}{4q+8} = \dfrac{2250}{4 + 8/q} < \dfrac{2250}{4} = \mathcal{L}$

That shows $r < \mathcal{L}$ for all $q > 0$.

(c) Assume a cost function $c(q)$. Suppose when 13 units are produced, the average cost is \$2.10 and the marginal cost is \$0.54 Find the approximate cost to produce 14 units. [4 points]

We know $a(q) = \dfrac{c(q)}{q}$ = average cost

$\therefore c(q) = a(q) \cdot q$

$c(13) = a(13) \cdot 13$

$= (2.10)(13)$

$= 27.30$

$c'(13) = 0.54$

We have that

$c(14) \approx c(13) + c'(13)$

$= 27.30 + 0.54$

$= \boxed{27.84}$

4. (a) If $y = 5x^3 + 4\sqrt{x+7}$ and $x = t^2 - \dfrac{4}{t^2}$ find $y'(1)$ where y' means differentiation with respect to t. [7 points]

$$y'(t) = \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$= \left[15x^2 + 2(x+7)^{-1/2}\right] \cdot \left[2t + 8t^{-3}\right]$$

When $t = 1$, $x = 1^2 - \frac{4}{1^2} = 1 - 4 = -3$

$$\therefore\ y'(1) = \left[15(-3)^2 + 2(-3+7)^{-1/2}\right]\left[2(1) + 8\right]$$

$$= [135 + 1][10] = \boxed{1360}$$

(b) How much would you have to invest (rounded up to the nearest cent) at 3.08% APR compounded semi-annually so that you would have the same amount at the end of one year as $10,000 invested at 3% APR compounding monthly ? [6 points]

Let desired amount be P. Solve the equation:

$$P(1.0154)^2 = 10{,}000(1.0025)^{12}$$

$$P = \frac{10{,}000(1.0025)^{12}}{(1.0154)^2}$$

$$\approx 9{,}993.97497$$

$\boxed{\text{Answer: } \$9{,}993.98}$

$$r_1 = \frac{3.08}{100} \times \frac{1}{2} = .0154$$

$$r_2 = \frac{3}{100} \times \frac{1}{12} = .0025$$

(c) Find $\displaystyle\lim_{x\to 0^-} \left(\frac{1}{|x|} + \frac{1}{x}\right)$. Be sure to show appropriate details in your answer. [4 points]

$x \to 0^-$ means $x \to 0$ and $x < 0$

$\therefore$ when $x < 0$, $|x| = -x$ so $\dfrac{1}{|x|} + \dfrac{1}{x} = -\dfrac{1}{x} + \dfrac{1}{x} = 0$

$$\therefore \lim_{x\to 0^-}\left(\frac{1}{|x|} + \frac{1}{x}\right) = \lim_{x\to 0^-} 0 = 0$$

$\boxed{\text{Answer} = 0}$

5. (a) For $x > 0$, let $F(x) = \dfrac{32 - 8\sqrt{x+7}}{x-9}$ when $x \neq 9$ and $F(9) = -1$ Determine whether F is continuous at $x = 9$ Sufficiently justify your answer. [8 points]

F is continuous @ $x=9$ if and only if $\lim_{x\to 9} F(x) = F(9)$. We check it is.

Firstly, $F(9) = -1$ as given above.

Next, $\lim_{x\to 9} F(x) = \lim_{x\to 9}\left[\dfrac{32 - 8\sqrt{x+7}}{x-9}\right]$

$$= \lim_{x\to 9}\left[\frac{32 - 8\sqrt{x+7}}{x-9} \cdot \frac{32 + 8\sqrt{x+7}}{32 + 8\sqrt{x+7}}\right]$$

$$= \lim_{x\to 9}\left[\frac{1024 - 64(x+7)}{(x-9)(32+8\sqrt{x+7})}\right]$$

$$= \lim_{x\to 9}\left[\frac{64(9-x)}{(x-9)(32+8\sqrt{x+7})}\right]$$

$$= -64 \cdot \lim_{x\to 9}\left[\frac{1}{8\sqrt{x+7}+32}\right] = (-64)\frac{1}{64} = -1$$

$$= F(9)$$

$\therefore$ F is continuous at $x = 9$.

(b) Let $u = (1+a)w_p - aw_c$ where w_p is a function of w_c and $a > 0$ is a constant. Show that $\dfrac{du}{dw_c} = (1+a)\left[z - b\right]$ where $z = \dfrac{dw_p}{dw_c}$ and $b = \dfrac{a}{1+a}$ [5 points]

$$\frac{du}{dw_c} = \frac{d}{dw_c}\left((1+a)w_p - aw_c\right)$$

$$= (1+a)\frac{dw_p}{dw_c} - a\frac{dw_c}{dw_c}$$

$$= (1+a)z - a$$

$$= (1+a)\left[z - \frac{a}{1+a}\right] = (1+a)\left[z - b\right].$$

(This page is intentionally left blank)

Some Midterm Test Statistics

Average $\approx 53.4\%$

$$\text{Pass \%} = \left(\frac{176-72}{176}\right) \times 100 \approx 59.1$$

(i.e. about 59.1% of the 176 students who wrote the test actually passed)

Approximate percentages of students earning a mark in the following deciles:

100 →	0%
90's →	0.6%
80's →	8.0%
70's →	13.6%
60's →	21.0%
50's →	15.9%
40's →	15.9%
30's →	11.9%
20's →	6.8%
10's →	5.7%
1's →	0.6%

Good, but a higher percentage of the 176 students should be $\geq 70\%$

25% of 176 is 44 students got a score $< 40\%$. That is very poor for UTSC.