University of Toronto at Scarborough
Department of Computer & Mathematical Sciences

MATA33S **Assignment 5 (3 Pages)** **Winter 2018**

Study: The pdf at our home page entitled "Notes on Determinants" is really important and quite straightforward. You should also Google "determinants" and look for some on-line notes and videos. Wikipedia gives useful, easy to understand ideas about determinants at the beginning of their notes. We will have lecture notes on determinants in Week 6 (Friday Feb 9 - Thursday Feb 15) and possibly before that, but there is no text material on them. This Assignment 5 and Solutions 5 will be helpful. Please carefully read the **Notes** on Page 3.

Terminology and Concepts to Learn: determinant of a 2×2 matrix, determinant by row and column expansion (especially the 3×3 and 4×4 case), determinant and transpose, determinant and product of matrices, determinant and row multiplication, determinant and its relation to invertibility, determinant calculation by row operations, Cramer's rule.

Problems:

1. In all parts of this question, use the 2×2 matrices A and B in Example 4 on page 249.

   (a) Find $det(A)$, $det(B)$, and verify that $det(AB) = det(A)det(B)$ and $det(A^T) = det(A)$

   (b) Verify that $det(A + B) \neq det(A) + det(B)$

   (c) Verify that $det(A^{-1}) = \dfrac{1}{det(A)}$

   (d) If C is the 2×2 matrix obtained by multiplying the first row of A by a number p and the second row of A by a number q, verify that $det(C) = pq(det(A))$.

   (e) Find all real numbers x for which the matrix $xI - A$ is invertible.

2. In all of this question, use the 3×3 matrices A and B given at the beginning of the Problems Section 6.3 on page 262.

   (a) Find $det(A)$, $det(B)$, and then verify that $det(AB) = det(A)det(B)$.

   (b) Verify that $|kA| = k^3|A|$ where k is a real number.

   (c) Let D be the 3×3 matrix obtained by interchanging rows 2 and 3 of A. Verify that $det(D) = -det(A)$.

3. Find $|M|$ where M is the 4×4 matrix whose rows are:

   $[2\ 0\ 3\ 1], [1\ 4\ 2\ 2], [-1\ 3\ 1\ 4]$, and $[0\ 2\ 1\ 0]$.

4. (a) Find a 3×3 matrix $A \neq 0$ such that $A^2 \neq 0$, but $A^3 = 0$.

   (b) Show that if B is an $n \times n$ matrix such that $B^k = 0$ for some natural number k, then B is not invertible.

   (Remark: Part (a) reminds us that if $B^k = 0$ it need not be the case that $B = 0$.)

5. Let P and Q be 5×5 matrices such that $det(P) = 2$ and $det(Q) = -3$.
   Find: (a) $det(PQ^2)$ (b) $det(3Q)$ (c) $det(-2(P^{-1}))$ (d) $det((3P)^{-1})$

6. Use Cramer's rule to find the solution to each of the following systems of equations:
   (a) Page 272, #13
   (b) Three equations in three variables: $x - y - 3z = -5$
   $$2x - y - 4z = -8$$
   $$x + y - z = -1$$

7. Imagine you are the buyer for a car rental agency. There are two types of cars to be bought: compact and sedan. Each compact car costs \$ c and each sedan costs \$ d, where $d > c$. You have a budget of \$ m and a total of n cars is to be purchased.

   (a) Let x and y represent the number of compact and sedans, respectively, that are bought subject to the specifications above. Write a system of equations that describe the specifications where the unknowns are x and y and then write the corresponding matrix equation for this system.
   (b) Use the formula for finding the inverse of a 2×2 matrix that involves the determinant to solve for x and y.
   (c) Use the method of reduction to solve for x and y.
   (d) Use Cramer's rule to solve for x and y (your answers to parts (b), (c), and (d) should be in terms of c, d, m, and n)
   (e) Verify that your results for x and y actually satisfy the equation you wrote in part(a).
   (f) Write a simple inequality in terms of c, d, m and n that must be true in order for the values of x and y to be meaningful in the context of this problem.
   (g) Even if the inequality you found in part (f) is true, why is it unlikely that the values of x and y would be meaningful in the context of this problem?

8. Verify by substitution that the stated solution for the system in Example 1 on page 273 actually satisfies the given system.

9. Determine the value(s) of the real number parameter s for which the system
   $$3sx - 2y = 4$$
   $$-6x + sy = 1$$
   has a unique solution. For these value(s) of s, use Cramer's rule to solve for x and y. (You should expect your answers to be expressed in terms of s). (Remark: Cramer's rule is especially useful in this problem).

10. This question deals with Problem 43 on page 284.

    (a) Let $x, y,$ and z represent the number of $D, E,$ and F shares bought, respectively. Show that the resulting system of equations describing the given information in the problem is

$$6x + 8y + 3z = 50,000$$
$$116x - 112y - 117z = 0$$
$$0x - 4y + z = 0$$

(b) Use Cramer's rule to solve for x, y, and z. (Cramer's rule is very useful in this problem because of the large values in the coefficient matrix of the system and because of the zeros on the right side of the system).

11. Suppose A is an $n \times n$ matrix such that the entries in each row add up to zero. Show that $det(A) = 0$. (Hint: Find the product AN where N is the $n \times 1$ column matrix each of whose entries is one. Think of what would happen if $det(A) \neq 0$.)

Notes:

1. **The Midterm Test is on Monday Feb 26, 5:00pm - 6:50pm.** A test information document will be posted and circulated in due course. You should expect that the test will cover all course material up to and including Week 6 (Friday Feb 9 - Thursday Feb 15) and all assignments/solutions relevant to that material.

2. **Quiz 3 is in Week 7 (Friday Feb 16 and Monday Feb 26 - Thursday March 1)**. Note that Quiz 3 will be written by many students in the same week that the test occurs. Here is the relevant material for Quiz 3:

   (a) Section 6.6 and related lecture notes (Matrix inverse). This is Problems 1, 2, 5 - 7 on Assignment 4

   (b) All of this Assignment 5 on determinants (see the pdf "Notes on Determinants" at our home page and do your own on-line learning and use lecture notes).

3. Remember: if you write quiz in a tutorial that is not an official one, then your score will be 0.

4. **Study Week is Monday Feb 19 - Friday Feb 23**. There are no MATA33 lectures or tutorials that week. Note that our test is on the first day back from Study week: Monday Feb 26, 5:00pm - 6:50pm.