University of Toronto at Scarborough
Department of Computer & Mathematical Sciences

MATA33S **Assignment 10 (3 Pages)** **Winter 2018**

Study: Sections 17.6 and 17.7 for this assignment. Read ahead in Section 17.9 and review techniques of integration as studied in MATA32. We omit Section 17.8. Also see the **Notes** below.

Terminology and Concepts to Learn Lagrange Multiplier method, constrained critical point, Lagrangian (i.e. Lagrange function).

Problems:

1. Section 17.7, Page 783 # 1, 2, 4, 5, 8, 9, 13, 14, 16, 20.
2. (a) Use the method of Lagrange multipliers to find the absolute maximum and minimum values of the function $f(x, y) = 2x - 3y + 5$ subject to the constraint $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$
   (b) Instead of using Lagrange multipliers, re-do part (a) by considering level curves of the function f that are tangent to the constraint curve.
3. A rectangular box without a lid is to be constructed from 12 m^2 of cardboard. Use Lagrange multipliers to find the maximum volume of such a box.
4. A rectangular plot of land of fixed area A is enclosed by a fence. The cost of fencing per horizontal unit of length is m and per vertical unit of length is n.
   (a) Use Lagrange multipliers to find the dimensions of the plot of land that can be enclosed by a fence of least cost and find this least cost.
   (b) If $m = n$, show that the plot of land of least cost fence enclosure is a square.
   (c) Solve part (a) using calculus of one variable (i.e. MATA32 ideas).
   (d) Show that there is no fencing of maximum cost (i.e. for the rectangular plot of fixed area A, show that you can find a fence enclosure whose cost is arbitrarily large).
5. Let $P = cx^a y^b$ represent production as a function of x and y where x represents the amount of labour and y represents the amount of capital (c, a, and b are positive constants, $a + b = 1$, and $x, y > 0$). Assume the unit labour cost is m and the unit capital cost is n and that we have a total budget of T (which is completely used). Use Lagrange multipliers to maximize production P subject to the budget constraint $mx + ny = T$. (Thus, you assume P has a maximum subject to the budget constraint and then use Lagrange multipliers to solve for the critical point(s) (x,y) for P. Enjoy this problem even more by re-reading # 6 on page 758 and by looking ahead to problem 12 below.)
6. Assume the function $f(x, y, z) = x + 2y + 3z$ has an absolute maximum and minimum subject to the constraints $x - y + z = 1$ and $x^2 + y^2 = 1$. Use Lagrange multipliers to find these values and where they occur.
7. In working through the parts of this question, you will gain some understanding as to why the method of Lagrange multipliers works.

(a) Assume $y = h(x)$ is a differentiable function of x and let $f(x, y) = C$ for some constant C. Use the chain rule to show that

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

(b) Use part (a) to conclude that the slope at each point (x, y) on the level curve $f(x, y) = C$ is given by $\frac{dy}{dx} = -\frac{f_x}{f_y}$

(c) Suppose we want to maximize $z = f(x, y)$ subject to the constraint $g(x, y) = 0$ (as in a typical Lagrange multiplier problem). Explain why $\frac{f_x}{g_x} = \frac{f_y}{g_y}$ Use this equation to justify why the Lagrange multiplier method works.

8. Section 17.7, Pages 783 - 784, # 10, 21 - 24.

9. Let $f(x, y, z, w) = x^2 + 2y^2 + 3z^2 - w^2$.

   (a) Find and classify the extrema of f.

   (b) Use algebra to verify your conclusion to part (a).

10. Does the function $f(x, y, z) = x^2 + y^2 + z^2 + (x^2 + y^2 + z^2)^{-1}$ have any extrema? If so, what are they and where are they located? (This is a slightly tricky problem)

11. The Silly Products Company (SPC) makes three versions of their best selling video game: the family version, the teen version, and the adult version. The profit on this game (in $\$1,000$) is given by $4x + 8y + 6z$ when x thousand family games, y thousand teen games, and z thousand adult games are produced.

    (a) Various costs (i.e. labour, manufacturing materials, legal, insurance, taxes, etc.) impost the constraint $x^2 + 4y^2 + 2z^2 = 800$. Find the maximum profit for the SPC on this video game.

    (b) Assume now that SPC can reduce the legal component of the costs and that this results in a new constraint $x^2 + 4y^2 + 2z^2 = 801$. Rework the problem in part (a) with this new (and slightly different) constraint. What does the solution tell you about the meaning of the Lagrange multiplier?

12. (a) The Cobb-Douglas production function for a manufacturer is given by $f(x, y) = 100x^{\frac{3}{4}}y^{\frac{1}{4}}$ where x represents the units of labour (at $\$150$ per unit) and y represents the units of capital (at $\$250$ per unit). The total cost of labour and capital is limited to $\$50,000$. Find the maximum production level for this manufacturer.

    (b) The Lagrange multiplier obtained in a production function is called the *marginal productivity of money.* Find the marginal productivity of money at the maximum production level in part (a).

Notes:

1. There will be one more posted Assignment/Solutions after this one.
2. The Final Exam is on Thursday, April 26, 9:00am - 11:50am. A exam information document and pre-exam office hour schedule will be posted and emailed in due course.