University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

FINAL EXAMINATION

MATA33 - Calculus for Management II

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Date: April 24, 2009
Duration: 3 hours

Provide the following information:

Surname (PRINT): ______________________________

Given Name(s) (PRINT): ______________________________

Student Number (PRINT): ______________________________

Signature: ______________________________

Read these instructions:

1. This examination paper has 14 numbered pages. It is your responsibility to ensure that at the beginning of the exam, all of these pages are included.
2. Put your solutions and/or rough work in the answer space provided. If you need extra space, use the back of a page or the blank page at the end of the exam.
3. You may use one standard hand-held calculator. All other electronic devices, extra paper, notes, and textbooks are forbidden at your workspace.

Print letters for the Multiple Choice Questions in these boxes:

1	2	3	4	5	6	7	8	9

Do not write anything in the boxes below.

A	1	2	3	4	5	6	7	8	**TOTAL**
45	14	12	14	12	12	14	12	15	150

Part A: Multiple Choice Questions For each of the following **print the letter of the answer you think is most correct in the boxes on the first page**. Each right answer earns 5 points and no answer/wrong answers earn 0 points. Justification is neither required nor rewarded, but a small workspace is provided for your calculations.

1. Assume the equation $z^2 = y - x$ defines z implicitly as a function of independent variables x and y. The value of z_{xx} at $(2, 3, 1)$ is

   (a) $-1/4$ (b) $1/4$ (c) $1/2$ (d) $-1/2$ (e) none of (a) - (d)

2. At the point $(0, 1)$ the function $f(x, y) = x^2 + y - e^y$ has

   (a) a relative minimum (b) a relative maximum (c) a saddle point
   (d) behavior that is not detected by the 2^{nd} derivative test (e) none of (a) - (d)

3. If $z = \ln(x + y^2) - y^2$ and $x = 3t$ and $y = 4t - 1$ then the value of $\dfrac{dz}{dt}$ at $t = 0$ is

   (a) -15 (b) -13 (c) 8 (d) 3 (e) none of (a) - (d)

4. The critical point of $f(x, y) = 8x^2 + y^2 + 5$ satisfying the constraint $8x - 2y = 12$ is

(a) $(1, -2)$ (b) $(\frac{12}{7}, \frac{6}{7})$ (c) $(-1, -10)$ (d) $(3, 6)$ (e) none of (a) - (d)

5. The value of $\int_0^3 \int_0^{\sqrt{9-x^2}} x \, dy \, dx$ is

(a) 18 (b) 9 (c) -9 (d) 27 (e) none of (a) - (d)

6. A system of n linear equations in n unknowns where $n > 1$

(a) has at least one solution
(b) has infinitely many solutions
(c) has a unique solution
(d) may not have any solutions
(e) can be solved by Cramer's rule
(f) both (c) and (e) are true

7. The function $G(x, y) = det \begin{bmatrix} x & x^{-1} \\ x & y \end{bmatrix}$

(a) has no critical point(s) (b) has a local maximum (c) has a local minimum

(d) has a saddle point (e) has none of properties (a) - (d)

8. If $A = \begin{bmatrix} 3 & 2 \\ 8 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $A + B^T = AC$ then C equals

(a) $\begin{bmatrix} 0 & 7 \\ 2 & 13 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 7 \\ 2 & -13 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & -7 \\ 2 & 13 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & -7 \\ -2 & 13 \end{bmatrix}$

(e) none of the above

9. Let R be the feasible region consisting of all points in and on the sides of the triangle with corners $(0, 0)$, $(1, 0)$, and $(2, 2)$. If $z = ax + by$ where a and b are positive constants then we may conclude that

(a) z has a maximum at a corner point of R

(b) there exist values of a and b for which z has a maximum at every point on some side of R

(c) the minimum value of z occurs at $(0, 0)$

(d) only (a) and (c) are true

(e) each of (a), (b), and (c) are true

(Check that you have printed the correct letter answers in first page boxes)

Part B: Full-Solution Questions Write clear and neat solutions in the answer spaces provided. Show all of your work. Full points are awarded for solutions only if they are correct, complete, and sufficiently display relevant concepts from MATA33.

1. In all of this question let $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$

   (a) Find the critical point(s) of f. For each one, use the 2^{nd} derivative test to determine whether it corresponds to a relative maximum, minimum, or a saddle point. [11 points]

   (b) Explain using relevant mathematical notation why f has no absolute extrema. [3 points]

2. A company makes two types of plastic toys: cars and trains. Each car results in a profit of \$6 and each train a profit of \$4. Each car requires 2 minutes on Machine A and 1 minute on Machine B and each train requires 1 minute on Machine A and 3 minutes on Machine B. In each production shift there are at most 3 hours available on Machine A, at most 5 hours available on Machine B, and at least 30 of each toy must be made.

   (a) Use the methods of linear programming to calculate how many of each toy should be made per production shift in order to maximize profit. Full marks are awarded for a correct solution that includes an accurate, labeled feasible region. [9 points]

   (b) By about what percentage will the maximum profit change (rounded to 2 decimal places) if we also require that equal numbers of cars and trains are made per shift ? [3 points]

3. (a) Let $z = 2u^2t^2 - 2u + 4t$ where $u = (y+3)\sqrt{x}$ and $t = e^{x-1}y^3$. Calculate the value of $\dfrac{\partial z}{\partial x}$ when $x = 1$ and $y = -2$. [7 points]

(b) Let $v = h(ar + bs, cr - ms)$ where $h(x, y)$ is a function for which the chain rule is valid and a, b, c, m are real numbers such that $am + bc = 0$. Prove that $mv_r + cv_s = 0$. [7 points]

4. A rectangular box has four sides, a top, and a base. The material costs are $\$6/m^2$ for the sides, $\$4/m^2$ for the top, and $\$14/m^2$ for the base. A budget of exactly $\$6K^2$ is used for the total material cost ($K > 0$ is a constant) for the box. Use the Lagrange multiplier technique to find the length, width, and height of the box of maximum volume subject to the constraint of the material cost above. (You may assume that the critical point obtained by the Lagrange multiplier technique actually does result in a maximum volume.) [12 points]

5. In all of this question, consider the "Cobb-Douglas" production function $P = Mw^{\alpha}c^{\beta}$ where w and c are independent variables and M, α, and β are positive constants and $\alpha + \beta = 1$.

   (a) Show that the marginal productivity with respect to w and the marginal productivity with respect to c are decreasing.

   [6 points]

   (b) Prove that $w^2 P_{ww} + c^2 P_{cc} = (-2\alpha\beta) P$ [6 points]

6. (a) Assume you have an annual income of E dollars and that x and y represent the amount (in dollars) of federal and provincial tax, respectively, on your income. The federal tax is 25% of that portion of your income that is left after the provincial tax has been paid. The provincial tax is 10% of that portion of your income after the federal tax has been paid. State the system of linear equations that describes the tax situation above and then solve for x and y as fractions of E in lowest terms. (i.e. find $x = \frac{a}{b}E$ and $y = \frac{c}{d}E$ where a, b, c, d are natural numbers and the fractions are in lowest terms). [8 points]

(b) Let A be an $n \times n$ matrix where $n > 1$. Assume the matrix equation $AX = B$ has a solution for every $n \times 1$ matrix B. Prove that A is invertible. [6 points]

7. A manufacturer has two products, A and B, that have hourly production levels of x and y units, respectively. The joint production cost function is $c(x,y) = \frac{3}{2}x^2 + 3y^2$. The demand for A is $a(x) = 60 - x^2$ and for B is $b(y) = 72 - 2y^2$. Calculate the production levels per hour that maximize profit. Do verify that your findings actually give maximum profit.

[12 points]

8. (a) Let $I = \int_0^1 \int_0^{2x} y\, dy\, dx + \int_1^3 \int_0^{-x+3} y\, dy\, dx$ (Thus, I is a sum of two double integrals.) By changing the order of integration, write I as a single double integral and evaluate it. Include a sketch of the region of integration.

[7 points]

(b) Evaluate $J = \int\int_R \frac{x\,e^y}{y}\,dA$ where R is the region between the curves $y = x$ and $y = x^2$ and $0 \leq x \leq 1$. [8 points]

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