*** SOLUTIONS ***

University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

Midterm Test

MATA33 - Calculus for Management II

Examiner: R. Grinnell

Date: June 17, 2011
Time: 7:00 pm
Duration: 120 minutes

Provide the following information:

(Print) Surname: SOLUTIONS

(Print) Given Name(s): ____________________

Student Number: ____________________

Signature: ____________________

Tutorial Number (e.g. TUT0033): ____________________

Carefully circle the name of your Teaching Assistant:

Zhuo (Joe) LI — Erik LOVBLOM — Yik Chau (Kry) LUI

Sujanthan (Suj) SRISKANDARAJAH

Read these instructions:

1. This test has ~~11~~ 10 numbered pages. It is your responsibility to check at the beginning of the test that all of these pages are included.
2. Answer all questions in the work space provided. If you need extra space, use the back of a page ~~or the last page,~~ and indicate clearly the location of your continuing work.
3. With the exception of the Multiple Choice questions, full points are awarded only for solutions that are correct, complete, and sufficiently display concepts and methods of MATA33.
4. You may use **one** standard hand-held calculator (graphing facility is permitted). The following are forbidden: laptop computers, Blackberrys, cell-phones, I-Pods, MP-3 players, extra paper, textbooks, or notes.
5. You are encouraged to write in pen or other ink, not pencil. Tests written in pencil will be denied any regrading privilege.

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Print letters for the Multiple Choice questions in these boxes.

1	2	3	4	5	6
C	C	E	D	F	C

Do not write anything in the boxes below.

Part A
18

Part B

1	2	3	4	5	6
13	18	14	10	15	12

Total
100

Part A - Multiple Choice

For each of the following **print the letter of the answer you think is most correct in the box at the top of page 2.** Each right answer earns 3 points and no answer or wrong answers earn 0 points. Justification is neither required nor rewarded, but a small workspace is provided for your calculations and rough work.

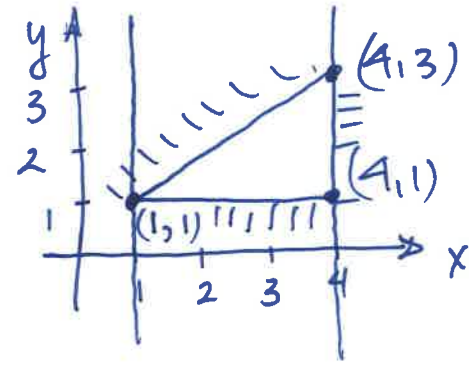
1. When x units of product **X** are sold and y units of product **Y** are sold the revenue is $F = x+2y$. If x and y satisfy the constraints $1 \leq x \leq 4$, $y \geq 1$, and $2x-3y+1 \geq 0$, then the maximum revenue is

   (A) 7 (B) 24 (C) 10 (D) 11 (E) none of (A) - (D)

$F(1,1) = 3$

$F(4,3) = 10$ ✓

$F(4,1) = 6$



2. If $A = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \\ 2 & 4 \end{bmatrix}$ then $2AB + B^T$ equals

   (A) $\begin{bmatrix} -9 & 15 \\ -2 & 50 \end{bmatrix}$ (B) $\begin{bmatrix} -5 & 0 \\ 13 & 26 \end{bmatrix}$ (C) $\begin{bmatrix} -9 & -2 \\ 15 & 50 \end{bmatrix}$ (D) $\begin{bmatrix} -10 & 0 \\ 16 & 54 \end{bmatrix}$

   (E) none of (A) - (D)

$$2\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 1 & 4 \end{bmatrix}$$

$$= 2\begin{bmatrix} -4 & -2 \\ 7 & 23 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} -8 & -4 \\ 14 & 46 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & -2 \\ 15 & 50 \end{bmatrix}$$

3. Assume G and H are 2×2 matrices such that $G^2 = H^2$. We must then conclude that

   (A) $G = H$ or $G = -H$ (B) $GH = HG$ (C) If $H = 0$ then $G = 0$

   (D) G and H have the same reduced form. (E) none of (A) - (D) need be true.

As a counter-example for (A), (B) and (D), consider $G = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $H = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

For (C), let $H = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $G = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Then $G^2 = H^2$, $H = 0$, yet $G \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

4. Let $U = [u_{ij}]$ be the 8×8 upper triangular matrix where $u_{ij} = -(i+j) - 2$ for all $i \leq j$. The largest element in U is

(A) -4 (B) -3 (C) -2 (D) 0 (E) none of (A) - (D)

$$U = [u_{ij}] = \begin{cases} -(i+j)-2 & i \leq j \\ 0 & i > j \end{cases}$$

$\because -(i+j)-2 \leq -4 \quad \forall i \leq j$, $u_{ij} \leq 0$ for all $1 \leq i,j \leq 8$. $\therefore$ max element $= 0$.

5. A real number x satisfying $det \begin{bmatrix} 3 & 2x \\ 3x & 4 \end{bmatrix} = 36$ is

(A) 0 (B) 2 (C) -2 (D) both 2 and -2 (E) a number not in (A) - (D)
(F) nonexistent

We have $12 - 6x^2 = 36$

$\longrightarrow -6x^2 = 24$

$\longrightarrow x^2 = -4$

which has no real solutions.

6. Exactly how many of the following statements are always true?

✓ • Every homogeneous system of $m \geq 2$ linear equations in $n > m$ variables has infinitely many solutions.

X • If a square matrix has no zero rows then, it is invertible. $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ not invertible

✓ • Different matrices of the same size can have the same reduced form. $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ & $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

X • If the product of two matrices P and Q equals the zero matrix, then at least one of P or Q must be the zero matrix. $\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$; neither $= 0$

X • A (non-zero) objective function defined on a non-empty feasible region has a maximum, minimum, or possibly both. Region need not be bounded

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4 (F) 5

(Be sure you have printed the Multiple Choice answers in the boxes on page 2)

Part B - Full Solution Problem Solving

1. For each of the following systems of linear equations, use the method of reduction to find the solution or determine that the system is inconsistent. If the system is consistent, be sure to display the reduced form of its augmented matrix.

(a)

$$3x - 8y - 14z = 3$$
$$x - 3y - 5z = 0$$
$$-4x + 11y + 19z = -3$$

[6 points]

$$\left(\begin{array}{ccc|c} 3 & -8 & -14 & 3 \\ 1 & -3 & -5 & 0 \\ -4 & 11 & 19 & -3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -3 & -5 & 0 \\ 3 & -8 & -14 & 3 \\ -4 & 11 & 19 & -3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -3 & -5 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & -1 & -3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -2 & 9 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Reduced form

$\boxed{\begin{array}{l} \text{Solution:} \\ x = 9 + 2r \\ y = 3 - r \\ z = r, \ r \in \mathbb{R} \\ r \text{ is a parameter} \end{array}}$

(b)

$$x_1 + 3x_2 - x_3 + 2x_4 = 1$$
$$-x_1 + 2x_2 + x_3 + 8x_4 = 9$$
$$x_1 + 8x_2 - x_3 + 12x_4 = 14$$
$$-x_1 + 7x_2 + x_3 + 18x_4 = 19$$

[7 points]

$$\left(\begin{array}{cccc|c} 1 & 3 & -1 & 2 & 1 \\ -1 & 2 & 1 & 8 & 9 \\ 1 & 8 & -1 & 12 & 14 \\ -1 & 7 & 1 & 18 & 19 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 3 & -1 & 2 & 1 \\ 0 & 5 & 0 & 10 & 10 \\ 0 & 5 & 0 & 10 & 13 \\ 0 & 10 & 0 & 20 & 20 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 3 & -1 & 2 & 1 \\ 0 & 5 & 0 & 10 & 10 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 5 & 0 & 10 & 10 \end{array}\right)$$

Row 3 yields
$0x_1 + 0x_2 + 0x_3 + 0x_4 = 3$
which has no sol$\underline{^n}$

$\boxed{\therefore \text{ given system has } \underline{\underline{\text{no}}} \text{ solution}}$

(We can also say that the original system is <u>inconsistent</u>)

2. Find the maximum and minimum values (and where they occur) for the objective function $Z = -5x + 10y$ subject to the four constraints:

$$2x - y \geq 0, \qquad 3x + 4y \leq 22, \qquad x - 2y \leq 4, \qquad 2x + 3y \geq 8$$

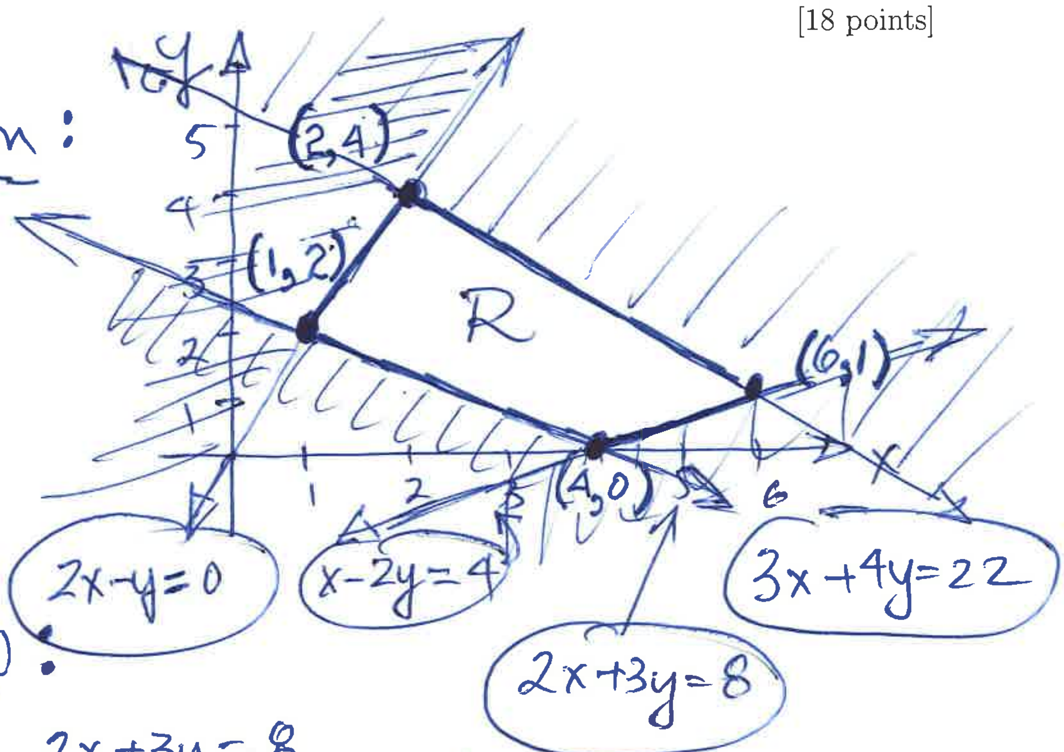
(To earn full points, your solution must include a neat, labeled diagram of the feasible region, the location of all point(s) where Z is optimized, and all calculations/justifications)

[18 points]

$R =$ Feasible region:

R is $\neq \phi$ and bounded

$\Rightarrow$ FTLP gives sol$^{\underline{n}}$.



Algebra details show:

$2x - y = 0$ crosses $2x + 3y = 8$ at $(1, 2)$

$2x - y = 0$ crosses $3x + 4y = 22$ at $(2, 4)$

$3x + 4y = 22$ crosses $x - 2y = 4$ at $(6, 1)$

$x - 2y = 4$ crosses $2x + 3y = 8$ at $(4, 0)$

Corner points:

$(2, 4)$

$(6, 1)$

$(4, 0)$

$(1, 2)$

By corner point evaluation (ie using FTLP)

$Z(2, 4) = 30$

$Z(6, 1) = -20$

$Z(4, 0) = -20$

$Z(1, 2) = 15$

$\Rightarrow$ MAX value of Z is 30 @ $(2, 4)$

$\Rightarrow$ MIN value of Z is -20 @ all points on segment joining $(4, 0)$ to $(6, 1)$

3. In all of this question let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -3 \\ 0 & -2 & 5 \end{bmatrix}$

(a) Use the method of row reduction to find A^{-1}. Show all of your work and correct notation for row operations. [9 points]

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & -2 & 5 & 0 & 0 & 1 \end{array}\right) \underset{(R_3+2R_2)\to R_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 2 & 1 \end{array}\right)$$

$$\sim (-R_3)\to R_3 \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -2 & -1 \end{array}\right)$$

$$\boxed{A^{-1} = \begin{pmatrix} 1 & 16 & 9 \\ 0 & -5 & -3 \\ 0 & -2 & -1 \end{pmatrix}}$$

$$\sim (R_2+3R_3)\to R_2 \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -5 & -3 \\ 0 & 0 & 1 & 0 & -2 & -1 \end{array}\right)$$

$$\sim (R_1-3R_3)\to R_1 \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 6 & 3 \\ 0 & 1 & 0 & 0 & -5 & -3 \\ 0 & 0 & 1 & 0 & -2 & -1 \end{array}\right)$$

$$\sim (R_1-2R_2)\to R_1 \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 16 & 9 \\ 0 & 1 & 0 & 0 & -5 & -3 \\ 0 & 0 & 1 & 0 & -2 & -1 \end{array}\right)$$

(b) Use your answer in (a) to solve for X in the matrix equation $BX = X + C$ where $B = \begin{bmatrix} 2 & 16 & 9 \\ 0 & -4 & -3 \\ 0 & -2 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ [5 points]

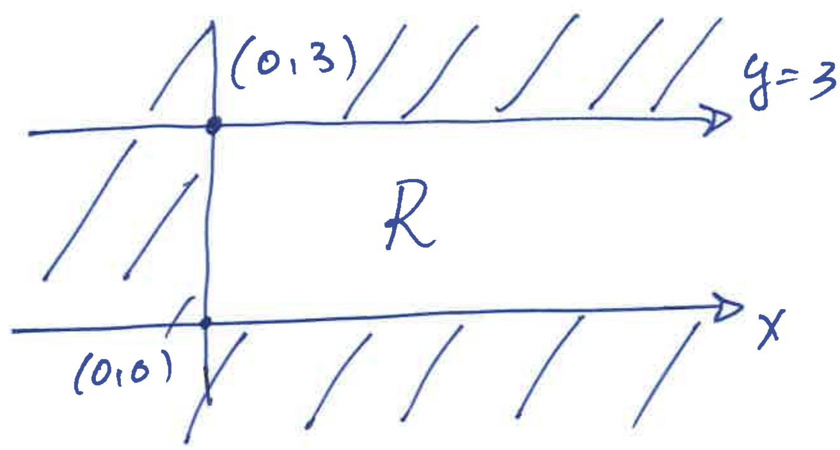
$$BX = X + C$$
$$\to BX - X = C$$
$$\to BX - IX = C$$
$$\to (B-I)X = C$$
$$\therefore B - I = A^{-1}$$

$$\to \therefore A^{-1}X = C$$
$$A(A^{-1}X) = AC$$
$$IX = AC$$
$$X = AC$$
$$= \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -3 \\ 0 & -2 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$
$$= \begin{pmatrix} 9 \\ -5 \\ 8 \end{pmatrix}$$

$$\boxed{X = \begin{pmatrix} 9 \\ -5 \\ 8 \end{pmatrix}}$$

4. Sketch the feasible region R defined by the two inequalities: $x \geq 0$ and $0 \leq y \leq 3$. Let $m < 0$ be a constant and consider the objective function $Z = mx - y$. Determine and discuss whether Z has a maximum and minimum on R. Full points will be awarded only if your answer is correct and mathematically well-justified. [10 points]

Feasible region R



① Z has no minimum on R as follows.

Points $(x, 0)$, $x \geq 0$, are in R

Evaluate: $Z(x, 0) = mx \longrightarrow -\infty$ as $x \to \infty$.

$\therefore$ Z has no minimum on R.

② Claim that the maximum value of Z is 0 at $(0,0)$.

Proof: $Z(0,0) = 0$ and $(0,0) \in R$.

Argument #1: Suppose $k > 0$ and we consider the line $mx - y = k$. We have $y = mx - k$

$\because$ $m < 0$ and the y-intercept is $-k < 0$, the line does not touch (or cross) R.

$\therefore$ if $(x,y) \in R$, then $Z = mx - y \leq 0$

so maximum is 0.

Argument #2: If $(x,y) \in R$, $mx - y \leq -y \leq 0$,

so maximum is 0. (tricky)

5. An investment company has two "balanced" portfolios: P_1 and P_2. Each portfolio consists of the same five sectors:

s_1 = energy, s_2 = financials, s_3 = industrials, s_4 = materials, s_5 = transportation.

Investment data is stored in the following matrix:

$$A = [a_{i,j}] = \begin{bmatrix} 9.0 & 9.3 & 7.1 & 5.6 & 9.0 \\ 8.6 & 9.9 & 6.8 & 6.2 & 8.5 \end{bmatrix}$$

where $a_{i,j}$ = the amount in \$1,000's invested in sector s_j for portfolio P_i. For example, $a_{1,3} = 7.1$ means that Portfolio 1 invests \$7,100 in sector s_3, which is industrials.

(a) State the matrices B, C, and F such that [7 points]

(i) the entries in AB^T give the total dollar amount invested in each portfolio.

$$B = 1{,}000 \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1000 & 1000 & 1000 & 1000 & 1000 \end{bmatrix}$$

(ii) the entries in CA give, for each sector, the average amount invested in \$1,000's over the two portfolios.

$$C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \end{bmatrix}$$

(iii) the entries in AF^T give, for each portfolio, the total amount invested in \$1,000's for the sector combination "energy - materials + 2(transportation)".

$$F = \begin{bmatrix} 1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

(b) Consider the following revised investment information. For both portfolios: there is a \$500 increase in energy investment; there is a 20% increase in financials investment; there is no change to the industrials investment; there is a \$1,000 decrease in materials investment; there is a 10% decrease in transportation investment. State the matrix R such that the entries in the sum $A + R$ reflect the revised investment information above all in units of \$1,000's. [5 points]

$$R = \begin{bmatrix} 0.5 & 1.86 & 0 & -1 & -.9 \\ 0.5 & 1.98 & 0 & -1 & -.85 \end{bmatrix}$$

Question 5 continued.

(c) What do the entries in the product AW represent, where $W = 2.5I$? [3 points]

2 possible answers.

Answer 1: Entries in AW represent the amount in units of \$2,500 invested in sector S_j for portfolio P_i $i=1,2$ $j=1,2,3,4,5$

Answer 2: Entries in AW represent a 150% increase (in each sector S_j for each portfolio P_i) from investments given in the matrix A.

6. (a) Let $A = \begin{bmatrix} 1 & 2 & 0 \\ -4 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Find all diagonal matrices D such that $D^2 = AA^T$. [5 points]

$$AA^T = \begin{bmatrix} 1 & 2 & 0 \\ -4 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -4 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 9 \end{bmatrix} = D^2$$

$\therefore$ D can be any of the 8 matrices given via $\begin{bmatrix} \pm\sqrt{5} & 0 & 0 \\ 0 & \pm 2\sqrt{5} & 0 \\ 0 & 0 & \pm 3 \end{bmatrix}$

(b) Let E be an arbitrary $n \times n$ matrix, $n \geq 2$, and let $C = EE^T$. If the diagonal entries in C are all 0, show that $E = 0$. [7 points]

$E = (e_{ij})$ $1 \leq i,j \leq n$ $\quad E = \begin{pmatrix} e_{11} & e_{12} & \cdots & e_{1n} \\ e_{21} & e_{22} & \cdots & e_{2n} \\ \vdots & \vdots & & \\ e_{n1} & e_{n2} & \cdots & e_{nn} \end{pmatrix}$

$\therefore E^T = (e_{ji})$ $1 \leq i,j \leq n$

Let $C = (c_{ij})$ $1 \leq i,j \leq n$. We have $c_{ii} = 0$ $1 \leq i \leq n$

For each i,

$$0 = c_{ii} = (e_{i1})^2 + (e_{i2})^2 + (e_{i3})^2 + \cdots + (e_{in})^2$$

$\therefore$ For each j, $(e_{ij})^2 = 0$ so $e_{ij} = 0$ for <u>all</u> $1 \leq i,j \leq n$

$\therefore E = 0$ as required.