Solution to Midterm , MATC34, 2001

1. (a) Series expansion for

$$f(z) = \frac{1+z}{1-z}$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

$$\frac{1+z}{1-z} = \sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} z^{n+1}$$

$$= 1 + 2\sum_{n=1}^{\infty} z^n$$

   (b) The radius of convergence of this series is 1 (because this is the radius of convergence of $\sum_{n=0}^{\infty} z^n$)

   (c) i. $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ is the Taylor series for $f(z) = \exp(z)$, for any z

   ii. $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ for all z with $|z| < 1$

2. $\int_\gamma f(z)dz =$

   (a) $f(z) = \frac{1-e^z}{z}$: this is holomorphic (it has a removable singularity at 0 so by Cauchy's theorem the integral is 0

   (b) $f(z) = \sin(z)$: the integral is 0 by Cauchy since f is holomorphic everywhere

   (c) $f(z) = \frac{z}{z-2}$: the integral is 0 by Cauchy since the only pole is at $z = 2$ which is outside $\gamma(0;1)$

3. $f(z) = \bar{z}$:

   $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \frac{\bar{h}}{h} = e^{-2i\theta}$x if $h = re^{i\theta}$

   So the limit as $h \to 0 \in \mathbf{C}$ does not exist.

4. Prove that if f holo. and $\mathrm{Re}(f)$ is constant then f is constant.

   If $f(z) = c$ when z is real, then $f(z) - c = 0$ on the real axis, a set with a limit point (for instance 0 is a limit point of the real axis). By the Identity Theorem $f(z) = c$ everywhere.

5. Compute the integral of the function $f(z)$

$$f(z) = \frac{1}{z^2 + z + 1}$$

about the following contours. All the contours should be traversed counterclockwise.

Solution:

$$z^2 + z + 1 = (z - A)(z - \bar{A})$$

where

$$A = \frac{-1 + \sqrt{3}i}{2} = e^{2\pi i/3}$$

We get

$$2\pi i \left(\frac{c_1}{z - A} + \frac{c_2}{z - \bar{A}} \right)$$

By partial fractions $c_1 = -c_2$. Also $-c\bar{A} + cA = 1$ so $c = \frac{1}{A - \bar{A}} = \sqrt{3}i$.

(a) γ is a rectangle with vertices -2, 2, $-2 + 2i$ and $2 + 2i$.
Solution: only A contributes so we get $2\pi ic$

(b) γ is a rectangle with vertices -2, 2, $-2 - 2i$ and $2 - 2i$.
Here only $\bar{A}$ contributs so we get $-2\pi i\bar{c}$

(c) γ is a circle with centre 0 and radius 2.
Both A and $\bar{A}$ contribute so the sum is 0