2 Curves in the complex plane

Definition 2.1 *A curve with parameter interval* $[\alpha, \beta]$ *is a continuous function* $\gamma : [\alpha, \beta] \to \mathbf{C}$*. It is called* closed *if* $\gamma(\alpha) = \gamma(\beta)$ *and* simple *if* $\alpha \leq s < t \leq \beta$ *implies* $\gamma(s) \neq \gamma(t)$ *for* $t - s < \beta - \alpha$ *(in other words the curve does not cross itself). It is called* smooth *if* γ *has continuous derivatives on* $[\alpha, \beta]$.

Definition 2.2 *A* path *is the union of finitely many smooth curves.*

Example 2.1

1. *If* $u, v \in \mathbf{C}$*, the line segment from* u *to* v *is* $\gamma(t) = (1-t)u + tv$.
2. *A circle of radius* r *traced counterclockwise around the point* a *in the complex plaine is*

$$\gamma(t) = a + re^{it}, \quad 0 \leq t \leq 2\pi.$$

Definition 2.3 *A complex-valued function* $h : [\alpha, \beta] \to \mathbf{C}$ *is* piecewise continuous *if there exist* $t_0 < t_1 < \ldots < t_n$ *with* $\alpha = t_0$ *and* $\beta = t_n$ *and continuous functions* h_k *on* $[t_k, t_{k+1}]$ *such that* $h(t) = h_k(t)$ *on* $[t_k, t_{k+1}]$ *. The function* h *need not be defined at* t_k.

Integral around a path α

1. Integration of a complex-valued function g on an interval $[\alpha, \beta]$, with $g(t) = u(t) + iv(t)$ where u and v are real-valued functions:

$$\int_a^b g(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt.$$

2. Integration along a path γ in the complex plane:

$$\int_\gamma g(z)dz = \int_a^b g(\gamma(t))\gamma'(t)dt$$

where $\gamma'(t) = \frac{d\gamma}{dt}$. This is the line integral from MATB42.

The join of the paths γ_1 and γ_2 is

$$\gamma(t) = \gamma_1(t), t \in [0, 1]$$

while

$$\gamma_2(t-1), t \in [1, 2]$$

(in other words, we concatenate the paths γ_1 and γ_2, the new path is γ_1 followed by γ_2)

Example 2.2 $\int_0^{2\pi} e^{it}dt = \int_0^{2\pi} \cos(t)dt + i\int_0^{2\pi} \sin(t)dt.$

Definition 2.4 *Let γ be a path with parameter interval $[a, b]$. Then $\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$.*

Example 2.3 $\int_\gamma (z-a)^n dz$ *where γ is a circle with radius r and centre a.* Solution: $\gamma(t) = a + re^{it}$ *so*

$$\int_\gamma (z-a)^n dz = \int_0^{2\pi} (re^{it})^n \Big(ire^{it}\Big)dt$$

$$-ir^{n+1}\int_0^{2\pi} e^{i(n+1)t}dt = 0$$

if $n \neq -1$ while the value of the integral is $2\pi i$ if $n = -1$.

Example 2.4 $\int_\gamma z^2 dz$ *where the integral is around a semicircle of radius R and centre 0 in the upper half plane.*

$$\int_\gamma z^2 dz = \int_0^1 \Big((2t-1)R\Big)^2 2Rdt + \int_0^\pi R^2 e^{2\pi it}(iRe^{it})dt$$
$$= 2R^3\Big(\frac{4}{3}t^3 - 2t^2 + t\Big)\ |_0^1 + \Big(\frac{1}{3}R^3 e^{3it}\Big)\ |_0^\pi$$
$$= 0$$

Fundamental theorem of calculus: Let $\gamma : [\alpha, \beta] \to \mathbf{C}$ be a path. Then if $F'(z)$ exists and is continuous on γ,

$$\int_\gamma F'(z)dz = F\Big(\gamma(\beta)\Big) - F\Big(\gamma(\alpha)\Big).$$

In particular, if γ is a closed curve, then $\int_\gamma F'(z)dz = 0$.

Proof 2.1 *Assume γ is smooth. Then $F \circ \gamma$ is differentiable on $[\alpha, \beta]$ with $(F \circ \gamma)'(t) = F'(\gamma(t))\gamma'(t)$. Then*

$$\int_\gamma F'(z)dz = \int_\alpha^\beta F'(\gamma(t))\gamma'(t)dt$$

$$= \int_\alpha^\beta (F \circ \gamma)'(t)dt$$

$$= \int_\alpha^\beta \mathrm{Re}(F \circ \gamma)'(t)dt + i\int_\alpha^\beta \mathrm{Im}(F \circ \gamma)'(t)dt$$

$$= \mathrm{Re}(F \circ \gamma)(t) + i\mathrm{Im}(F \circ \gamma)(t) \mid_\alpha^\beta = F(\gamma(\beta)) - F(\gamma(\alpha)).$$

More generally choose $\alpha = t_0 < t_1 < \ldots < t_n = \beta$ so that γ is smooth on $[t_i, t_{i+1}]$.

Theorem 2.5 (Estimation Theorem) *Let γ be a path with parameter interval $[\alpha, \beta]$. Let $f : \gamma \to \mathbf{C}$ be continuous. Then $| \int_\gamma f(z)dz| \leq \int_\alpha^\beta | f(\gamma(t))\gamma'(t) | dt$.*

Proof 2.2 *For $g : [\alpha, \beta] \to \mathbf{R}$ integrable, we have*

$$| \int_\alpha^\beta g(t)dt | \leq \int_\alpha^\beta | g(t) | dt.$$

So

$$e^{-i\phi} | \int_\gamma f(z)dz | = | \int_\alpha^\beta f(\gamma(t))\gamma'(t)dt | e^{-i\phi} = \int_\alpha^\beta f(\gamma(t))\gamma'(t)dt$$

for some $\phi \in \mathbf{R}$. So $| \int_\gamma f(z)dz | = \int_\alpha^\beta \mathrm{Re}\Big(e^{i\phi} f(\gamma(t))\gamma'(t)\Big)dt$. Apply to

$$g(t) = \mathrm{Re}(e^{i\phi} f(\gamma(t))\gamma'(t)).$$

Hence

$$| \int_\gamma f(z)dz | \leq \int_\alpha^\beta \mathrm{Re}\Big(e^{i\phi} f(\gamma(t))\gamma'(t)\Big)dt.$$