University of Toronto at Scarborough
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MAT C34F 2018/19

Complex Variables

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Assigment 1

1. A number $z = x + iy$ in the complex plane can be written in polar coordinates as
$$z = re^{i\theta}$$
where $r \geq 0$ is a real number $r = \sqrt{x^2 + y^2}$ and $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ where $\cos(\theta) = x/r$ and $\sin(\theta) = y/r$.

   The equation $z^n = 1$ has n roots: these are
$$z = e^{2\pi i m/n}$$
where $m = 0, \ldots, n - 1$.

2. The *complex conjugate* of z is $\bar{z} = x - iy$.

3. The *modulus* of z is $|z| = \sqrt{x^2 + y^2}$.

4. The logarithm of a complex number $z = re^{i\theta}$ is
$$[\mathrm{Log}(z)] = \{\log(r) + i\theta : \ \theta \in [\mathrm{Arg}(z)]\}$$
where the argument of z is

   $[\mathrm{Arg}(re^{i\theta})] = \{\theta + 2\pi n\}$ for $n = 0, \pm 1, \pm 2, \ldots$

1. A complex-valued function f is *differentiable* at z if

$$\frac{df}{dz}(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists.

Here, the limit is taken as the complex number h tends to zero. If one considers h tending to zero along a fixed direction in the complex plane (in other words $h = re^{i\theta}$ where $r \to 0$ but θ remains constant) the limit must give the same value regardless of the value of the angle θ.

2. A complex valued function f is *holomorphic* at z_0 if f is differentiable at all z in an open set containing z_0.

3. If a function $f(z) = u(x, y) + iv(x, y)$ is differentiable at z, then it satisfies the *Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

WARNING: it is not true that IF a function satisfies the Cauchy-Riemann equations THEN it is differentiable at z_0. What is true is:

Theorem: If $f(z) = u(x, y) + iv(x, y)$ satisfies the Cauchy-Riemann equations and the partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ exist in a neighbourhood of z_0 and are continuous at z_0, then f is differentiable at z_0.

4. Conditions showing that a function is holomorphic:

   (a) $f(z) = z$ is holomorphic

   (b) if f and g are holomorphic, so is fg

   (c) if f and g are holomorphic and $g(z) \neq 0$, then f/g is holomorphic at z

   (d) if f and g are holomorphic then the composition $f(g(z))$ is a holomorphic function of z (using the Chain Rule for complex functions)

(e) A complex power series

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

defines a holomorphic function inside its radius of convergence. Furthermore, the function obtained by differentiating a complex power series term by term

$$g(z) = \sum_{n=0}^{\infty} n c_n (z-a)^{n-1}$$

has the same radius of convergence as the power series for f, and equals the derivative $f'(z)$.

5. Examples of holomorphic functions:

   (a) polynomials

   (b) the exponential function $f(z) = e^z$, defined by

$$e^z = \sum_{n=0}^{\infty} z^n/n!$$

   (this series converges for all values of z)

   (c) trigonometric functions

$$\cos(z) = (e^{iz} + e^{-iz})/2$$

   and

$$\sin(z) = (e^{iz} - e^{-iz})/(2i)$$

Assignment 2

1. Contour integrals along a path $\gamma : [a,b] \to \mathbf{C}$ in the complex plane with parameter interval $[a,b]$ are defined by

$$\int_{\gamma} f(z)dz = \int_a^b f(\gamma(t))\frac{d\gamma}{dt}dt.$$

2. Integrals along a path are independent of the parametrization of the path.

3. The Fundamental Theorem of Calculus asserts that if F is defined on an open set containing a path γ with parameter interval $[a, b]$ and the derivative $F'(z)$ exists and is continuous at every point of γ, then

$$\int_\gamma F'(z)dz = F(\gamma(b)) - F(\gamma(a)).$$

4. $$\int_\gamma z^n dz = 0 \text{ if } n \neq -1; \quad = 2\pi i \text{ if } n = -1.$$

5. **Estimation Theorem:** If γ is a path with parameter interval $[a, b]$ and the function f is continuous on γ, then

$$\left| \int_\gamma f(z)dz \right| \leq \int_a^b |f(\gamma(t))\gamma'(t)|dt.$$

6. **Theorem on interchange of summation and integration:** Suppose that γ is a path and $U, u_0, u_1, \ldots$ are continuous complex-valued functions on γ and $\sum_{k=0}^\infty u_k(z)$ converges to $U(z)$ for all z in γ, and $|u_k(z)| \leq M_k$ for some M_k with $\sum_{k=0}^\infty M_k < \infty$. Then

$$\sum_{k=0}^\infty \int_\gamma u_k(z)dz = \int_\gamma \left(\sum_{k=0}^\infty u_k(z) \right) dz = \int_\gamma U(z)dz.$$

7. **Region:** A region is a connected open set.

8. **Homotopy:** two curves are homotopic in a region G if one can be deformed into the other while staying entirely within G.

9. **Simply connected:** A region G is simply connected if every closed path can be deformed to a point, while staying entirely in G.

10. **Jordan curve theorem:** Every closed path γ in the complex plane separates the plane into an inside $I(\gamma)$ which is bounded and an outside $O(\gamma)$ which is unbounded.

11. **Indefinite Integral Theorem:** Let f be a continuous complex valued function on a convex region G, with the property that the integral of f around any triangle in G is 0. Then there is a holomorphic function F for which
$$F' = f.$$

12. **Antiderivative Theorem:** A holomorphic function f on a convex region has an antiderivative (in other words a function F for which $F' = f$).

13. **Cauchy's theorem:** If f is holomorphic inside and on a closed contour γ, then $\int_\gamma f(z)dz = 0$.

14. **Deformtion theorem:** If γ is a positively oriented contour and f is holomorphic inside and on γ (except possibly at $z = a$), then
$$\int_\gamma f(z)dz = \int_{\gamma(a;r)} f(z)dz$$
where a is a point inside γ and $\gamma(a;r)$ is the circular contour with centre a and radius r, for r so small that $\gamma(a;r)$ lies inside γ

15. **Logarithm:** If G is any open region not containing 0, then the logarithm can be defined as follows:
$$\log(z) - \log(a) = \int_\gamma \frac{1}{w} dw,$$
where γ is a path with parameter interval $[0,1]$ contained entirely in G with endpoints $z = \gamma(1)$ and $a = \gamma(0)$.

16. **Winding number:** The winding number of a closed path γ around a point w is defined as
$$n(\gamma, w) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - w} dz.$$
Informally, this is the number of times γ winds around w. For example the winding number of the counterclockwise unit circle around the point $w = 0$ is $n(\gamma, 0) = 1$.

Assignment 3

1. Cauchy's integral formula
2. Taylor's theorem
3. Zeroes of holomorphic functions
4. Identity theorem
5. Maximum modulus theorem
6. Liouville's theorem

Assignment 4: Singularities

1. Laurent's theorem
2. Singularities:
   (a) Removable singularity
   (b) Pole
   (c) Essential singularity
      i. isolated
      ii. non-isolated

   Assignment 5: Residues

      i. *Residue:* If f is a meromorphic function then the residue of f at a is the coefficient of $1/(z-a)$ in the Laurent series of f at a. The residue of f at a is written as $res\{f(z); a\}$.
      ii. *Cauchy's residue formula:* If f is holomorphic inside and on a positively oriented contour γ except for a finite number of poles at $a_1, \ldots, a_m$ inside γ, then

$$\int_\gamma f(z)dz = 2\pi i\Big(\sum_{k=1}^{m} res\{f(z); a_k\}\Big).$$

iii. *Zero-pole theorem* Let f be holomorphic inside and on a positively oriented contour γ except for P poles inside γ (counted according to their orders). Let f be nonzero on γ and have N zeros inside γ (counted according to their orders). Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = N - P.$$

iv. *Rouché's theorem* Let f and g be holomorphic inside and on a contour γ and suppose $|f(z)| > |g(z)|$ on γ. Then f and $f+g$ have the same number of zeros inside γ.

v. *Calculation of residues:* If

$$f(z) = \frac{g(z)}{(z-a)^m}$$

for some positive integer m, where g is holomorphic at a, then

$$res\{f(z);a\} = \frac{1}{(m-1)!}g^{(m-1)}(a).$$

In particular, if $f(z) = \frac{g(z)}{(z-a)}$ where g is holomorphic at a then

$$res\{f(z);a\} = g(a).$$

If

$$f(z) = \frac{g(z)}{h(z)}$$

where g and h are holomorphic at a, where $g(a) \neq 0$, $h(a) = 0$ and $h'(a) \neq 0$ then

$$res\{f(z);a\} = \frac{g(a)}{h'(a)}.$$

vi. *Estimation of integrals*

A. Basic inequalities: If $z_1, \ldots z_n$ are any complex numbers, then

B. $|z_1 + z_2| \leq |z_1| + |z_2|$

C. $|z_1 + \ldots + z_n| \leq |z_1| + \ldots + |z_n|$

D. $|z_1 + z_2| \geq \Big||z_1| - |z_2|\Big|$

E. $|z_1 + \ldots + z_n| \geq |z_1| - |z_2| - \ldots - |z_n|$

F. $|z_1| \leq |z_2| \Longleftrightarrow 1/|z_1| \geq 1/|z_2|$

G. If f is a continuous function on a path γ with parameter interval $[\alpha, \beta]$ then

$$\int_\gamma f(z)dz \leq \int_\alpha^\beta \Big|f(\gamma(t))\gamma'(t)\Big|dt$$

H. *Jordan's inequality:* If $0 < \theta \leq \pi/2$, then

$$\frac{2}{\pi} \leq \frac{\sin\theta}{\theta} \leq 1.$$

I. *Large arc estimate:* If γ is a circular arc $\gamma(\theta) = Re^{i\theta}$ (for $\theta_1 < \theta < \theta_2$) then

$$\Big|\int_\gamma f(z)dz\Big| \leq \int_{\theta_1}^{\theta_2} \Big|f(Re^{i\theta})\Big|Rd\theta.$$

J. *Small arc estimate*: If f has a simple pole of residue b at the point a and f is holomorphic on some punctured disc around a (except at the point a), then letting

$$\gamma_\epsilon(\theta) = a + \epsilon e^{i\theta}$$

for $\theta_1 \leq \theta \leq \theta_2$ (this is an arc of radius ϵ and centre a that passes through the angles from θ_1 to θ_2) then

$$\lim_{\epsilon \to 0} \int_{\gamma_\epsilon} f(z)dz = ib(\theta_2 - \theta_1)$$

In particular if $\theta_1 = 0$ and $\theta_2 = 2\pi$ then

$$\lim_{\epsilon \to 0} \int_{\gamma_\epsilon} f(z)dz = 2\pi ib.$$

WARNING: this estimate can only be used if the pole at a is a SIMPLE pole.

Assignment 5: Applications of Contour Integrals

i. *Integrals over the real line or the positive real axis:*

Let f be a function on the real line, which extends to a meromorphic function F on the upper half plane which has no zeros or poles on the real line.

Complete the integral along the real line to a contour integral by adding a semicircular contour of radius R.

A. The contour integral can now be evaluated by using residues.

B. To compute the integral over the real line, one must show that the integral around the semicircle of radius R tends to 0 as $R \to \infty$. (Use the basic inequalities in the last part of chapter 7.)

C. Sometimes, care must be taken to choose an appropriate function F whose restriction to the real line is f, in order that the integral of F over the semicircle tends to 0 as $R \to \infty$.

D. At times it is more convenient to compute the integral of a complex valued function whose real part is the integral we want. For example $e^{iz} = \cos(z) + i\sin(z)$, and its behaviour on a semicircle at infinity makes it easier to use the Large Arc Estimates than for either $\cos(z)$ or $\sin(z)$. So to compute $\int_{-\infty}^{\infty} \frac{\sin(x)}{x}$ we would use contour integrals to compute $\int_{-\infty}^{\infty} \frac{e^{ix}}{x}$, and then take the imaginary part.

ii. *Integrals where the function has a pole along the real axis:* In this case it is necessary to modify the contour by cutting out a small arc of radius ϵ around the pole. If the pole is a simple pole, use the Small Arc Estimate to obtain the value of the integral around the small arc in the limit as $\epsilon \to 0$. If the pole is not a simple pole, modify the function F so that it has a simple pole at the point in question.