MATC34 2013 Solutions to Assignment 2

1. $f(z) = \frac{z+2}{z} = 1 + \frac{2}{z}$

   Find $\int_C f(z)dz$ when $C =$ (i)

   $$C = \{z = 2e^{i\theta}, 0 \le \theta \le \pi\}$$

   The integral of 1 is

   $$\int dz = \int_0^\pi 2e^{i\theta} i d\theta = 2e^{i\theta} \mid_0^\pi = 2(-2) = -4$$

   The integral of $\frac{2}{z}$ is

   $$\int 2\frac{dz}{z} = 2\int_0^\pi i d\theta = 2\pi i$$

   so the integral is $-4 + 2\pi i$.

   (ii) $C = \{z = 2e^{i\theta}, 0 \le \theta \le 2\pi\}$ The integral of 1 is $2e^{i\theta}|_0^{2\pi} = 0$. The integral of $\frac{2}{z}$ is $2\int_0^{2\pi} i d\theta = 4\pi i$. so the total is $4\pi i$. The contour C is a square with vertices $0, 1, i, 1+i$.

2. To conclude that $\int_C (3z+1)dz = 0$, we can use Cauchy's theorem because the function $f(z) = 3z+1$ is holomorphic inside C. Alternatively we can parametrize the curve C as the disjoint union of $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ (where γ_1 is the line segment from 0 to 1, so $\gamma_1(t) = t$) γ_2 is the line segment from 1 to $1+i$, ($\gamma_2(t) = 1 + it$) γ_3 is the line segment from $1+i$ to i ($\gamma_3(t) = 1 + i - t$)

   γ_4 is the line segment from i to 0) ($\gamma_4(t) = i(1-t)$)

   So

   $$\int_{\gamma_1} (3z+1)dz = \int_0^1 (3t+1)dt = 3/2 + 1$$

   $$\int_{\gamma_2} (3z+1)dz = \int_0^1 (3(1+it)+1)d(it) = \int_0^1 (3+3it+1)idt = 4i - 3/2$$

   $$\int_{\gamma_3} (3z+1)dz = \int_0^1 3(1+i-t)+1)(-dt) = \int_0^1 (4+3i-3t)(-dt) = -4-3i+3/2$$

   $$\int_{\gamma_4} (3z+1)dz = \int_0^1 (3i(1-t)+1)(-idt) = \int_0^1 (3-i-3t)dt = 3-i-3/2.$$

   Total

   $$= 3/2 + 1 - 3/2 + 3/2 - 3/2 - 4 + 3 + 4i - 3i - i = 0.$$

3. (a)
$$\gamma(t) = 1 + ie^{it}, 0 \le t \le \pi$$
Semicircle centre 1 radius 1 angle $\pi/2$ to $3\pi/2$

   (b) Join of $[-1,1], [1,1+i], [1+i,-1-i]$
   The polygonal path is as shown

   (c) $\gamma(t) = e^{it}, t \in [0,\pi]$
   $e^{-it}, t \in [\pi, 2\pi]$
   A unit semicircle in the upper half plane centre with centre 0 traversed first counterclockwise and then clockwise.
   The curve is closed.
   (i) is simple
   (i), (ii), (iii) are paths
   (i) is smooth

4. (a) $\int_{\gamma(0;1)} |z|^4 dz = \int_{\gamma(0;1)} dz = 0$

   (b) $\int_{\gamma(0;1)} (\mathrm{Re}(z))^2 \, dz = \int_0^{2\pi} \cos^2\theta i e^{i\theta} d\theta = i \int_0^{2\pi} (\cos^3\theta + i\cos^2\theta\sin\theta) d\theta$
   Now $\int_0^{2\pi} \cos^3\theta d\theta = 0$ and also $\int_0^{2\pi} \cos^2\theta \sin\theta d\theta = -\int w^2 dw$ (where $w = \cos\theta$) $= -\cos^3\theta/3 \mid_0^{2\pi} = 0$. So $\int_{\gamma(0;1)} \mathrm{Re}(z)^2 dz = 0$.

   (c) $\int_\gamma \frac{z^4-1}{z^2} dz = \int_\gamma (z^2 - z^{-2}) dz = 0$ since $\int_{\gamma(0;1)} z^n dz = 0$ unless $n = -1$.

   (d) $\int_\gamma \sin(z) dz = -\cos(z)|0^{2\pi} = 0$.

   (e)
$$\int_\gamma z^{-1}(\bar{z} - 1/2) dz = \int_0^{2\pi} e^{-2i\theta} i e^{i\theta} d\theta - 1/2 \int QQQQ dz/z$$
$$= i \int e^{-i\theta} d\theta - 1/2 \int dz/z = 0 - 1/2(2\pi i) = -i\pi$$

5. If f is holomorphic and real valued in a region (a connected open set), then f is constant.
   $f = u + iv$, where $v = 0$
$$u_x = v_y = 0$$
   so $u_x = 0$.
$$u_y = -v_x = 0$$

so $u_y = 0$. Hence $u_x = u_y = 0$ so u is constant on any connected open set in the domain. (u need not be constant everywhere: for example let $f(x, y) = -1$ when $x < -1$ and $+1$ when $x > 1$. f is holomorphic on the open set $\{(x, y) : |x| > 1$ but not constant.

6. (i) $\int_{\gamma(0;2)} \frac{1}{1+z^2} dz$

The function $f(z) = \frac{1}{1+z^2}$ has poles at $z = +i$ and $z = -i$.

$$f(z) = \frac{1}{(z+i)(z-i)}$$

Replace the contour by two semicircular contours Γ_1 and Γ_2.

$$\int_{\gamma(0;2)} f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz$$

Each of the integrals around Γ_1 and Γ_2 can be evaluated by the deformation theorem:

$$\int_{\Gamma_1} \frac{1}{(z+i)(z-i)} dz = \int_{\gamma(i;r)} \frac{1}{(z+i)(z-i)} dz = \frac{1}{2i} \int \gamma(i;r) \frac{dz}{z-i} = \frac{2\pi i}{2i}$$

Similarly

$$\int_{\Gamma_2} \frac{1}{(z+i)(z-i)} dz = \int_{\gamma(-i;r)} \frac{1}{(z+i)(z-i)} dz = -\frac{1}{2i} \int \gamma(i;r) \frac{dz}{z+i} = \frac{2\pi i}{-2i}$$

So $\int_{\gamma(0;2)} f(z)dz = 0$.

(ii) $\int_{\gamma(3i,\pi)} \frac{1}{1+z^2} dz$

The integrand fails to be holomorphic at $z = \pm i$. $\gamma(3i; \pi)$ has centre $3i$ and radius π.

$3i - i = 2i; |3i - i| = 2 < \pi$

$3i + i = 4i; |3i + i| = 4 > \pi$

So the only point inside the contour where the integrand fails to be holomorphic is $z = +i$.

By the deformation theorem,

$$\int_{\gamma(3i;\pi)} \frac{1}{z^2+1} dz = \int_{\gamma(i;\epsilon)} \frac{1}{(z-i)(z+i)} dz = \frac{2\pi i}{2i} = \pi.$$

7. All possible values of $\int_\gamma \frac{1}{1+z^2} dz$ where γ is a path starting at 0, ending at 1 and not passing through $\pm i$.

One path is the path along $\mathbf{R}$; $\gamma(t) = t, 0 \leq t \leq 1$

This integral is

$$\int_0^1 \frac{1}{1+t^2} dt = \tan^{-1}(t) \mid_0^1 = \pi/4.$$

Integrals along all other paths will differ from this one by $\int_\Gamma \frac{1}{1+z^2} dz$ where Γ is a closed path starting at 0.

We can decompose any such path into a closed path Γ_1 enclosing only $+i$ and a closed path Γ_2 enclosing only $-i$. The integral around Γ_1 (if it is simple) is

$$\int_{\Gamma_1} \frac{1}{(z-i)(z+i)} dz = \int_{\gamma(i;\epsilon)} \frac{1}{(z-i)(z+i)} dz = \frac{2\pi i}{2i} = \pi.$$

Similarly if the path Γ_1 winds n times around $+i$, (where n could be negative if Γ is negatively oriented), the integral is $n\pi$. Likewise the integral around Γ_2 is $-\pi m$ if Γ_2 winds m times around $-i$ (where m is negative if Γ_1 is negatively oriented). Thus the possible values of the integral around Γ are πn ($n \in \mathbf{Z}$) so the possible values of the integral around a path γ from 0 to 1 are $\pi/4 + \pi n$ ($n \in \mathbf{Z}$).