MATC34 2013 Solutions to Assignment 3

1. $\gamma = \{|z-i| = 2\}$. So we need to compute $\int_\gamma g(z)dz$ where $g(z) = \frac{1}{z^2+4} = \frac{1}{(z-2i)(z+2i)}$. The pole at $z = 2i$ is inside γ, while that at $z = -2i$ is outside γ. So by the Cauchy integral formula, the integral is $2\pi i f(2i)$ where $f(z) = \frac{1}{z+2i}$. Thus the integral is $2\pi i f(2i)$ where

$$f(z) = \frac{1}{z+2i}.$$

Thus the integral is $\frac{2\pi i}{4i} = \pi/2$.

2. $f(z) = \sin^2 z$, expand as $\sum_n c_n z^n$.

$$f(z) = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 = -\frac{1}{4}\left(e^{2iz} + e^{-2iz} - 2\right)$$

$$= -\frac{1}{4}\left(\sum_{n\geq 0} \frac{2^n i^n z^n}{n!} + \sum_{m\geq 0} \frac{2^m(-i)^m z^m}{m!} - 2\right)$$

Only even m and n contribute; odd m and n cancel out. So

$$f(z) = -\frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{2^{2n}(-1)^n z^{2n}}{(2n)! - 2}\right)$$

The radius of convergence is ∞.

3. If there were a function f holomorphic on $D(0;1)$ for which $f(1/n) = 0$ when n is even and $f(1/n) = 1/n$ when n is odd, since the sequence $\{1/2n : n = 1, 2, 3, \ldots\}$ has a limit point in the disc, by the identity theorem the function must be zero everywhere. Thus it cannot take the value $1/n$ at the point $z_n = 1/n$ when n is odd. So no such function exists.

4. f holomorphic on $D(0;1)$ Consider $h(z) = \overline{f(-\bar{z})}$ Claim h is holomorphic on $D(0;1)$ Let $f(z) = u + iv$ $z = x + iy$ implies $\bar{z} = x - iy$ and $-\bar{z} = -x + iy$.

So $f(-\bar{z}) = u(-x, y) + iv(-x, y)$

$\bar{f}(-\bar{z}) = u(-x, y) - iv(-x, y) := U(x, y) + iV(x, y)$

so $U(x,y) = u(-x,y)$ and $V(x,y) = -v(-x,y)$.

To show that h is holomorphic on $D(0;1)$ it suffices to show that

(a) U and V satisfy the Cauchy-Riemann equations

(b) U and V have continuous first order partial derivatives with respect to x and y

(b) follows because u and v have continuous first order partial derivatives w.r.t. x and y

Proof of (a): Define $u_x = \frac{\partial u}{\partial x}(x,y)$.

Thus

$$\frac{\partial}{\partial x} u(-x,y) = -\frac{\partial}{\partial(-x)} u(-x,y) = -u_x(-x,y).$$

$$\frac{\partial U(x,y)}{\partial x} = \frac{\partial}{\partial x} u(-x,y) = -u_x(-x,y).$$

$$\frac{\partial V(x,y)}{\partial y} = -\frac{\partial}{\partial y} v(-x,y) = -v_y(-x,y).$$

So since $u_x = u_y$, $\partial U/\partial x = \partial V/\partial y$.

$$\frac{\partial U(x,y)}{\partial y} = \frac{\partial}{\partial y} u(-x,y) = u_y(-x,y)$$

$$-\frac{\partial V}{\partial x} = -\frac{\partial}{\partial x}\left(-v(-x,y)\right) = -v_x(-x,y).$$

So since $u_y = -v_x$,

$$\frac{\partial U(x,y)}{\partial y} = -\frac{\partial V(x,y)}{\partial x}$$

Thus (U,V) satisfy the Cauchy-Riemann equations.

Thus h is holomorphic.

Thus $g(z) = f(z) - \overline{f(-\bar{z})}$ is holomorphic. If f is real on the imaginary axis, $g(z) = 0$ for z on the imaginary axis, so by the identity theorem $g(z) = 0$ for z in $D(0;1)$. So $u(x,y) + iv(x,y) - u(-x,y) + iv(-x,y) = 0$ if and only if $u(x,y) = u(-x,y)$ and $v(x,y) = -v(-x,y)$, equating the real and imaginary parts.

5. $G = \{z : \ |\text{Re}(z)| < 1 \text{ and } |\text{I}m(z)| < 1\ \}$. f is continuous in the closure of G (which is obtained by replacing $<$ by $\leq$ in the two inequalities above). f is also holomorphic in G, and $f(z) = 0$ if $\text{Re}(z) = 1$. Define g by

$$g(z) = f(z)f(iz)f(-z)f(-iz).$$

We can see that $g(z) = 0$ on the whole boundary of G. This is true because:

- $f(z) = 0$ if $\text{Re}(z) = 1$ (by definition)
- $f(-z) = 0$ if $\text{Re}(-z) = 1$, in other words if $\text{Re}(z) = -1$
- $f(iz) = 0$ if $\text{Re}(iz) = 1$, in other words if $\text{Im}(z) = 1$ So $f(iz) = 0$ if $\text{Im}(z) = 1$
- $f(-iz) = 0$ if $\text{Re}(-iz) = 1$ so $f(-iz) = 0$ if $\text{Im}(z) = -1$

Thus by the Maximum Modulus Theorem, the maximum of $|g|$ occurs on the boundary of G, so $g = 0$ on $\tilde{G}$, and $f = 0$ on $\tilde{G}$.

6.

$$\int_{gamma(0;1)} \frac{dz}{(z-a)(z-b)}$$

Case 1: If $|a| < 1, |b| < 1$, both a and b lie inside the contour. By Cauchy integral formula, the integral equals

$$= 2\pi i \left(\frac{1}{b-a} + \frac{1}{a-b} \right) = 0$$

Case 2:If $|a| < 1, |b| > 1$ then

$$\int_{\gamma(0;1)} \frac{dz}{(z-a)(z-b)} = \int_{\gamma(0;1)} \frac{dz}{(z-a)} f(z)$$

for $f(z) = \frac{1}{z-b} = \frac{2\pi i}{a-b}$.

(similar result if $|a| > 1$ and $|b| < 1$)

Case 3: If $|a| > 1$, $|b| > 1$ then the integral is 0.