University of Toronto at Scarborough
Division of Physical Sciences, Mathematics

MAT C34F 2000/2001

Final Examination

Friday, December 15, 2000 ; 9:00 - 12:00

No books or calculators may be used.

You may use any theorems stated in class, as long as you state them clearly and correctly.

(1) **(15 points)** Compute

$$\int_\Gamma \frac{\sin(z)dz}{(2z-\pi)^2(z-2\pi)}$$

where Γ is the circle with centre 0 and radius π.

Solution: Rewrite as

$$\int_\Gamma \frac{\sin(z)dz}{4(z-\pi/2)^2(z-2\pi)}$$

The only pole that contributes is at $z=\pi/2$. The residue at that pole is $h'(\pi/2)$ where

$$h(z) = \frac{\sin(z)}{4(z-2\pi)}$$

According to Cauchy's integral formula for derivatives (text 13.9 p. 157) this is

$$h'(\pi/2) = (\frac{1}{4}\left(\frac{\cos(z)}{(z-2\pi)} - \frac{\sin(z)}{(z-2\pi)^2}\right)_{z=\pi/2}$$

$$= -\frac{1}{4(\pi/2-2\pi)^2} = -\frac{1}{9\pi^2}.$$

Hence the integral is

$$2\pi i h'(\pi/2) = -2\pi i/(9\pi^2) = -2i/(9\pi).$$

(2) **(15 points)** Find the singularities of the following functions. For each singularity, state whether it is a removable singularity, a pole or an essential singularity, and justify your statement. Identify the orders of all poles.

You need not consider whether the point at infinity is a singularity.

(a)

$$\frac{1}{(z^2+4)\sin(\pi z)}$$

Solution: Singular at $z=\pm 2i$: pole of order 1

Singular at z an integer: pole of order 1

(b)

$$\frac{z}{\sin(\pi z)}$$

Singular at z: an integer : pole of order 1 (if $z \neq 0$, removable singularity $z=0$.

(c)

$$\sin(1/z)$$

Singular at $z = 0$: essential singularity

(3) **(15 points)**

(a) Using residues, find the integral

$$\int_\gamma \frac{e^{z^3} dz}{(z-1)^2}$$

where γ is the circle with centre 0 and radius 3.

Solution: The only poles are at $z = 1$. Let $f(z) = e^{z^3}$ so $f'(z) = 3z^2 e^{z^3}$. Then the residue at $z = 1$ is $f'(1) = 3e$ and the integral is $6\pi i e$.

(b) Let Γ be the square with vertices $+1+i$, $-1+i$, $-1-i$ and $+1-i$ traversed in that order (in other words counterclockwise). Compute the integral

$$\int_\Gamma \frac{\cos(z) dz}{z^3}.$$

Solution: The only residue is at $z = 0$. Because $\cos(z) = 1 - z^2/2 + \ldots$, the residue at 0 is $-1/2$ and the integral is $-\pi i$.

(4) **(15 points)**

(a) Find the Laurent series at 0 for

$$\frac{\sin(z)}{z^2}.$$

Solution: $1/z - z/3! + \sum_{n=2}^{\infty} z^{2n+1}/(2n+3)!$

(b) Find the first three nonzero terms in the Laurent series at 0 for

$$\frac{z^2}{\sin(z)}.$$

Solution:

$$\frac{z^2}{z(1 - z^2/3! + z^4/5! + \ldots}\,)$$

Write $x = z^2$

Use

$$\frac{1}{1 + ax + bx^2} = 1 - (ax + bx^2) + \frac{(ax + bx^2)^2}{2} + \text{terms of order } x^3.$$

The sum is $1 - ax + x^2(-b + a^2)$. Use $a = -1/3!$, $b = 1/5!$

Then the first terms are $z + (1/6)z^3 + (-1/120 + (1/36))\, z^5$.

(5) **(15 points)** Find a Möbius transformation

$$f(z) = \frac{az+b}{cz+d}$$

for which

$$f(0) = 0$$
$$f(1) = \infty$$
$$f(\infty) = 1$$

Find the image under f of the real axis and the imaginary axis. If the image is a line, you should give the line in terms of the direction perpendicular to it and a point on it. If the image is a circle, you should state the centre and the radius of this circle.

Solution:
$b = 0$
$c + d = 0$
$a = c$
so $a = c = -d, b = 0$ and

$$f(z) = \frac{z}{z-1}.$$

Three points on the real axis are $0, 1, \infty$
These points are sent to $0, \infty, 1$ so the image of the real axis is the real axis.
Three points on the imaginary axis are $0, i, \infty$
These are sent to

$$0, 1, \frac{i}{i-1} = \frac{1}{\sqrt{2}} e^{-i\pi/4} = \frac{i(-i-1)}{2} = \frac{1-i}{2}.$$

The circle through these three points is the circle with centre 1/2 and radius 1/2.

(6) **(15 points)** Use contour integrals to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{x^4 + x^2 + 1}.$$

Solution:
Write $x^2 = y$

$$y^2 + y + 1 = (y + w)(y + \bar{w})$$

where $w = \frac{1}{2} + \frac{\sqrt{3}}{2} i = e^{i\pi/3}$. Also denote $v = e^{i\pi/6}$, which satisfies $w = v^2$.
So

$$x^4 + x^2 + 1 = (x^2 + w)(x^2 + \bar{w}) = (x^2/v^2 + 1)(x^2/\bar{v}^2 + 1)$$
$$= (x/v + i)(x/v - i)(x/\bar{v} + i)(x/\bar{v} - i)$$

$$= (x + iv)(x - iv)(x + i\bar{v})(x - i\bar{v}).$$

The argument given in class shows that the integral over the real axis is equal to the limit as $R \to \infty$ of the contour integral around the semicircle of centre 0 and radius R. (because the degree of $x^4 + x^2 + 1$ is 4, which is greater than or equal to 2). So we get a contribution from the poles at $iv = e^{2i\pi/3}$ and $i\bar{v} = e^{i\pi/3}$.

The residue at iv is

$$\frac{1}{(2iv)(iv + i\bar{v})(iv - i\bar{v})}$$

The residue at $i\bar{v}$ is

$$\frac{1}{(i\bar{v} + iv)(i\bar{v} - iv)(2i\bar{v})}$$

The integral is $2\pi i$ times the sum of these two residues.

Solution:

(7) **(10 points)** Which of the following statements are true, and which are false? If a statement is true because of a theorem stated or proved in class, you should give the name of the theorem or state it (you are not required to prove it). If a statement is not true, you should give an example which shows it isn't valid.

: (1.) If $\int_{\triangle} f dz = 0$ for all triangles $\triangle$ in a region G, then f is holomorphic in G.
Answer: True (Cauchy's theorem for triangles)

: (2.) If f is a holomorphic function on the complex plane, then f is bounded.
False ($f = exp$ is holomorphic every where on the complex plane but it is not bounded)

: (3.) If f is a bounded holomorphic function on the complex plane, then f is constant.
True (Liouville's theorem)

: (4.) If γ is a simple closed curve and $\int_{\gamma} f dz = 0$ then f is holomorphic inside γ.
False: if $f(z) = 1/z^2$ and γ is the unit circle, f is not holomorphic at 0, but the integral of f around the unit circle is 0.

: (5.) If f is holomorphic inside a simple closed curve γ, then $\int_{\gamma} f dz = 0$
True (Cauchy's theorem)

: (6.) If f is holomorphic on the unit disc $\{z \mid |z| \leq 1\}$ then the minimum value of $|f|$ occurs on the boundary $\{z \mid \ |z| = 1\}$.
False (for $f(z) = z$ the minimum absolute value occurs at $z = 0$)

: (7.) If f is holomorphic on the unit disc $\{z \mid |z| \leq 1\}$ then the maximum value of $|f|$ occurs on the boundary $\{z \mid \ |z| = 1\}$.
True (maximum modulus theorem)

: (8.) There is a complex number z for which $\cos^2(z) + \sin^2(z) = 2$
False: If there were, then

$$e^{iz} + e^{-iz})^2 - (e^{iz} - e^{-iz})^2 = 8$$

which means

$$2 + 2 = 8$$

which is false.

: (9.) The integral $\int_{\gamma} \frac{1}{z} dz$ has the same value for any simple closed curve γ for which $\gamma(t)$ is not equal to 0 for any t.
False : the integral depends on the winding number of γ around 0

: (10.) It is possible to define a continuous holomorphic function f on a region containing the unit circle $\{z| \ |z| = 1\}$ such that $f(z)$ is equal to one of the values of $\sqrt{z}$ everywhere on the unit circle.

False: $f(z) = \sqrt{z}$ is not continuous on the unit circle, because if $\sqrt{z}$ were a continuous function along a path counterclockwise from $z = 1$, we would have $f(e^{(2\pi-\epsilon)i})$ tends to -1 as $\epsilon \to 0$, but $e^{(2\pi-\epsilon)i} \to 1$ as $\epsilon \to 0$ and $f(1) = 1$.