University of Toronto at Scarborough
Division of Physical Sciences, Mathematics

MAT C34F 2001/2002

<u>Final Examination</u>

Friday, December 14, 2001 ; 2:00–5:00

No books or calculators may be used.

You may use any theorems stated in class, as long as you state them clearly and correctly.

1. **(20 points)** At which values are the following functions singular? You need not consider whether or not the point at infinity is a singular point.

   Identify the type of singularity (pole, removable singularity, essential singularity).

   (i)
$$\frac{1}{(z-1)\sin^2(z)}$$

   (ii)
$$\frac{e^{1/z}}{z^2}$$

   (iii)
$$\frac{z}{e^z - 1}$$

2. **(15 points)**

   Compute
$$\int_\gamma \frac{e^{2z}}{(z-1)^3}$$
   where γ is a circle of radius 2 and centre 0 oriented counterclockwise.

3. **(15 points)** (i) State Liouville's theorem.

   (ii) Suppose f is holomorphic for all z in the complex plane. Suppose also that the k-th derivative $f^{(k)}(z)$ is bounded (in other words there is some number C such that $|f^{(k)}(z)| \leq C$ for all z).

   Show that f is a polynomial of degree less than or equal to k.

4. **(15 points)** (i) Compute the Laurent series of

$$\frac{z}{(z+2)^2}$$

at $z = 0$.

(ii) Give the three terms corresponding to the lowest powers of z in the Laurent series at 0 of the following function:

$$f(z) = \frac{1}{e^z - 1}$$

5. **(15 points)** Use residue calculus to compute the following integral:

$$\int_{-\infty}^{\infty} \frac{dx}{x^4 + x^2 + 1}$$

6. **(20 points)**

(i) Find a Möbius transformation

$$f(z) = \frac{az + b}{cz + d}$$

for which

$$f(0) = 2$$
$$f(1) = i$$
$$f(\infty) = 0$$

(ii) Let $u(x, y) = e^x \cos(y)$. Show that u is harmonic, and find a real-valued function $v(x, y)$ for which $u(x, y) + iv(x, y)$ is a holomorphic function of $x + iy$.