University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

MAT C34F 2018/19

Problem Set #2

Due date: Thursday, October 4, 2018 at the beginning of class

1. Let C be the perimeter of the square with vertices at the points $z = 0$, $z = 1$, $z = 1+i$ and $z = i$ traversed once in that order.

   (a)Compute
$$\int_C \bar{z}^2 dz.$$

   Solution:
$$\begin{aligned} &\int_0^1 x^2 dx + \int_0^1 (1-iy)^2 i dy + \int_1^0 (x-i)^2 dx + \int_1^0 (-iy)^2 i dy \\ &= 1/3 + \int_0^1 (1-2iy-y^2) i dy - \int_0^1 (x^2-2ix-1)dx + \int_0^1 y^2 i dy \\ &= 1/3 + i(1-i-1/3) - (1/3-i-1) + 1/3i \\ &= 1/3 + 2/3i + 1 + 2/3 + i + 1/3i \\ &= 2+2i \end{aligned}$$

   (b) Show that
$$\int_C e^z dz = 0.$$

   Solution: e^z is holomorphic everywhere, so we may use Cauchy.

2. (a) If $P(z)$ is a polynomial and Γ is any closed contour, explain why $\int_\Gamma P(z)dz = 0$.

   Solution: A polynomial has an antiderivative (in other words there exists another polynomial Q with $P(z) = dQ/dz$. This means
$$\int_\gamma P(z)dz = \int_\gamma dQ/dz dz = Q(1) - Q(0) = 0$$
   (since $Q(0) = Q(1)$)

   (b) Explain why part (a) shows that the function $f(z) = 1/z$ has no antiderivative in the punctured plane $\mathbf{C} - \{0\}$.

   Solution: If f had an antiderivative, then the argument in part (a) would show that $\int_\gamma f(z)dz = 0$. However we showed in class that for the unit circle this integral is $2\pi i$.

3. Show that if C is a positively oriented circle and z_0 lies outside C , then

$$\int_C \frac{dz}{z - z_0} = 0.$$

Solution: $f(z) = \frac{1}{z-z_0}$ is holomorphic inside and on the circle C (because z_0 is outside C). Hence we may apply Cauchy.

4. For each curve C and function f find the value of

$$\int_C f(z)dz$$

:

$$f(z) = \frac{z+2}{z} = 1 + 2/z$$

and C is

(a) the semicircle $z = 2e^{i\theta}$ $(0 \leq \theta \leq \pi)$

Solution:

$$\int_0^{\pi} 2ie^{i\theta} d\theta + 2\int_0^{\pi} 2id\theta = 2e^{i\theta}|_0^{\pi} + 4i\pi = 2(-1-1) + 4i = -4 + 4i$$

(b) the circle $z = 2e^{i\theta}$ $(0 \leq \theta \leq 2\pi)$

Solution: This is $2\int dz/z = 4\pi i$ (by earlier calculation)

5. Show that if C is the boundary of the square with vertices at the points $z = 0$, $z = 1$, $z = 1 + i$, $z = i$ and the orientation of C is counterclockwise, then

$$\int_C (3z + 1)dz = 0.$$

Solution: The function $f(z) = 3z+1$ is holomorphic everywhere, so we may use Cauchy to show that the integral

$$\int_C f(z)dz = 0.$$