University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

MAT C34F 2018/19

Problem Set #3

Due date: Tuesday, October 23, 2018 at the beginning of class

(1) Evaluate

$$\int_\Gamma \frac{z dz}{(z+2)(z-1)}$$

where Γ is the circle $|z| = 4$, clockwise.

Solution:

Note that

$$\frac{1}{(z+2)(z-1)} = \frac{A}{z+2} + \frac{B}{z-1} = \frac{A(z-1)+B(z+2)}{(z+2)(z-1)}$$

This means

$$A = -B$$

and

$$-A + 2B = 1 = -3A$$

so

$$A = -\frac{1}{3}.$$

So our integral is

$$\frac{1}{3}\int_\Gamma z dz \left(-\frac{1}{z+2} + \frac{1}{z-1}\right)$$

The integral is

$$\frac{1}{3}\int_\Gamma \left(-\frac{(z+2)-2}{z+2} + \frac{(z-1)+1}{z-1}\right) dz$$

$$= \frac{2}{3}\int_\Gamma \frac{dz}{z+2} + \frac{1}{3}\int_\Gamma \frac{dz}{z-1}.$$

By Cauchy's integral formula, this is

$$\left(\frac{2}{3}(2\pi i) + \frac{1}{3}(2\pi i)\right)(-1)$$

The minus sign is because the integral is clockwise. If it had been counterclockwise, the answer would be $+2\pi i$.

$$= -2\pi i.$$

(2) (a) Evaluate

$$\int_{\Gamma_+} \frac{2z^2 - z + 1}{(z-1)^2(z+1)}$$

where Γ_+ is

$$\Gamma_+(t) = 1 + e^{-it}, 0 \leq t \leq 2\pi$$

Γ_+ is a circle with radius 1 and center 1, clockwise.

Solution:

Let

$$f(z) = \frac{2z^2 - z + 1}{(z-1)^2(z+1)}$$

We use (partial fractions)

$$\frac{1}{(z-1)^2(z+1)} = \frac{Az+B}{(z-1)^2} + \frac{C}{(z+1)}$$

$$= \frac{(Az+B)(z+1) + C(z-1)^2}{(z-1)^2(z+1)}$$

$$= \frac{(A+C)z^2 + (A+B-2C)z + B + C}{(z-1)^2(z+1)}$$

It follows that

$$A = -C,$$
$$A + B - 2C = 0,$$
$$B + C = 1$$

Solving, we find

$$3C - B$$
$$3C + C = 1$$
$$C = \frac{1}{4}$$

Hence

$$\frac{1}{(z-1)^2(z+1)} = \frac{-\frac{1}{4}z + \frac{3}{4}}{(z-1)^2} + \frac{1}{4(z+1)}$$

$$= \frac{-\frac{1}{4}(z-1) - \frac{1}{4} + \frac{3}{4}}{(z-1)^2} + \frac{1}{4(z+1)}$$

$$= -\frac{1}{4(z-1)} + \frac{1}{2(z-1)^2} + \frac{1}{4(z+1)}.$$

So
$$\int_{\Gamma_+} f(z)dz) = -\frac{1}{4}(-1)\int_{\Gamma}\frac{dz}{z-1}$$
(the minus sign is because the integral is clockwise)
$$= \frac{1}{4}(2\pi i) = \pi i/2$$
.

This is because
$$\int_{\Gamma_+}\frac{dz}{(z-1)^2} = 0$$
by our earlier results about the integral
$$\int_{z:|z|=1}\frac{dz}{z^n} = 0$$
unless $n = 1$.

(b) Evaluate
$$\int_{\Gamma_-}\frac{2z^2 - z + 1}{(z-1)^2(z+1)}$$
$$\Gamma_-(t) = -1 + e^{it}, 2\pi < t < 4\pi$$
(anticlockwise) $\Gamma + -$ is a circle with radius 1 and center -1, anticlockwise.

Solution: By similar reasoning
$$\int_{\Gamma_-} f(z)dz = \int_{\Gamma_-}\frac{dz}{4(z+1)} = \frac{1}{4}(2\pi i) = \pi i/2.$$

(3) Compute
$$\int_{|z|=2}\frac{dz}{z^2+z+1}$$
Solution:

Let γ be the oriented curve $\{z||z| = 2\}$ oriented counterclockwise.
$$\int_{\gamma}\frac{dz}{z^2+z+1} = \int_{\gamma}\frac{dz}{(z-a_+)(z-a_-)}$$
where
$$z^2 + z + 1 = (z-a_+)(z-a_-)$$

so
$$a_\pm = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i = e^{\pm 2\pi i/3}.$$
Both a_+ and a_- are inside γ.

Let γ_+ be semicircle in the upper half plane with center 0 and radius 2. Let γ_- be a semicircle in the lower half plane with center 0 and radius 2. Let both γ_+ and γ_- be oriented counterclockwise.

The integral of f around γ is the sum of the integral around γ_+ and the integral around γ and $[2, -2]$. (The two integrals over the oriented line segment $[-2, 2]$ in the real axis sum to zero, because these segments have opposite orientations.)

By the Cauchy integral formula,
$$\int_\gamma \frac{f(z)dz}{z - a_+} = 2\pi i f_+(a_+)$$
(where $f_+(z) = \frac{1}{z - a_-}$)
$$= 2\pi i \frac{1}{a_+ - a_-} = \frac{2\pi i}{\sqrt{3} i}.$$

Similarly
$$\int_{\gamma_-} \frac{f_-(z)dz}{z - a_-} = 2\pi i f_-(a_-)$$
(here we have $f_-(z) = \frac{1}{z - a_+}$ which is holomorphic inside γ_-).

Hence by Cauchy integral formula,
$$\int_{\gamma_-} f_-(z) = 2\pi i f_-(a_-) = 2\pi i \frac{1}{(a_- - a_+)} 3i = 2\pi i (\frac{1}{-\sqrt{3} i}.$$
So
$$\int_\gamma f(z)dz = 0,$$
since this is the sum of
$$\int_{\gamma_+} \frac{f_+(z)dz}{z - a_+} + \int_{\gamma_-} \frac{f_-(z)dz}{z - a_-}.$$

(4) Compute
$$\int_{|z|=2} \frac{\sin(z)dz}{z^2 + 1}$$

Solution:

$$\int_{|z|=2} \frac{\sin(z)dz}{z^2+1} = \int_{|z|=2} \sin(z)dz()i/2)(\frac{1}{z+i} - \frac{1}{z-i}$$

By the Cauchy integral formula,

$$\int_{|z|=2} \frac{\sin(z)dz}{z+i} = 2\pi i \sin(-i)$$

and

$$\int_{|z|=2} \frac{\sin(z)dz}{z-i} = 2\pi i \sin(i).$$

So the integral is

$$(i/2)(2\pi i)(\sin(-i) - \sin(i)) = 2\pi \sin(i) = \pi i(e - e^{-1})$$

using

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

(5) Let $f(z) = \frac{1}{z+1}$. Find an expansion

$$f(z) = \sum_{n=0}^{\infty} c_n (z-i)^n$$

which is valid in a disk with center i and radius r.
Solution:

$$f(z) = \frac{1}{z+1} = \frac{1}{(z-i)+(1+i)} = \frac{1}{(1+i)(1+\frac{z-i}{1+i}}$$

$$= \frac{1}{1+i} \sum_{n=0}^{\infty} (-1)^n (\frac{z-i}{1+i})^n.$$

This expansion is valid when $|\frac{z-i}{1+i}| < 1$, in other words when $|z-i| < |1+i| = \sqrt{2}$. So the disc radius is $r < \sqrt{2}$.

(6) Suppose f is a function which is holomorphic everywhere on the complex plane and satisfies

$$f(z+1) = f(z)$$
$$f(z+i) = f(z)$$

for all z. Prove that f is constant.
Solution: Any holomorphic function satisfying these conditions is bounded, because there is M for which $|f(z)| \leq M$ for all $z = x + iy$ where $0 \leq x \leq 2\pi$ and $0 \leq y \leq 2\pi$. This is true

because a continuous function on a compact set is bounded. Then $|f(z)| \leq M$ for all z in the complex plane, since for all z there is some $w = a + ib$ with $0 \leq a \leq 2\pi$ and $0 \leq b \leq 2\pi$ and some integers m, n such that $z = w + m + ni$. So $f(z) = f(w)$, so $|f(z)| = |f(w)| \leq M$. Thus f is a bounded function which is holomorphic everywhere on the complex plane. By Liouville's theorem, f must be constant.

(7) Is there a holomorphic function with $f(1/n) = 1$ when n is even) and $f(1/n) = -1$ when n is odd. ? If so, exhibit the function. If not, give a proof.

Solution: Since 0 is a limit point of $\{1/n, n \text{ even }\}$, any such function must equal 1 everywhere, by the Identity Theorem. suBut 0 is also a limit point of the set of $1/n$ when n is odd, so by the same argument any such function must equal -1 everywhere. This is a contradiction. So no such function can exist.