University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

MAT C34F 2018/19

Problem Set #4

Due date: Thursday, November 15, 2018 at the beginning of class

(1) Find the Laurent expansion of

$$(z^2-1)^{-2}$$

valid

(a) for $0 < |z-1| < 2$

Solution: For $0 < |z-1| < 2$,

$$\frac{1}{(z^2-1)^2} = \frac{1}{(z-1)^2((z-1)+2)^2} = \frac{1}{4(z-1)^2}(1+(\frac{z-1}{2})))^{-2}$$

Put $w = \frac{z-1}{2}$.

Then, since $\frac{1}{1+w} = \sum_{n=0}^{\infty}(-1)^n w^n$ for $|w| < 1$,

$$\frac{d}{dw}(1+w)^{-1} = -(1+w)^{-2} = \sum_{n=1}^{\infty}(-1)^n n w^{n-1}$$

So

$$\frac{1}{(1+\frac{z-1}{2})^2} = \sum_{n=1}^{\infty}(-1)^{n-1} n \left(\frac{z-1}{2}\right)^{n-1}$$

and

$$\frac{1}{4(z-1)^2}\left(1+\frac{z-1}{2}\right)^2 = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{n(z-1)^{n-3}}{2^{n+1}}$$

(b) for $|z+1| > 2$

Solution: In this case

$$(z^2-1)^{-2} = \frac{1}{(z+1)^2(z+1-2)^2} = \frac{1}{(z+1)^4(1-\frac{2}{z+1})^2}$$

$$= \frac{1}{(z+1)^4}\sum_{n=1}^{\infty}(-1)^n n\left(\frac{2}{z+1}\right)^n$$

$$= -\sum_{n=1}^{\infty}\frac{2^{n-1}}{(z+1)^{n+3}}$$

(2) Find the principal part of the Laurent expansion of $(e^z - 1)^{-2}$ around 0.

Solution:

$$e^z - 1 = \sum_{n=1}^{\infty} \frac{z^n}{n!} = z(1 + z/2 + z^2/3! + \dots)$$

$$= z(1 + A(z))$$

where

$$A(z) = \sum_{n=2}^{\infty} \frac{z^{n-1}}{n!}$$

So

$$(e^z - 1)^{-2} = \frac{1}{z^2}\left(\frac{1}{1 + A(z)}\right)^2$$

and

$$\frac{1}{1 + A(z)} = 1 - A(z) + A(z)^2 - \dots$$

so

$$\left(\frac{1}{1 + A(z)}\right)^2 = \left(\frac{1}{1 + w}\right)^2$$

where $w = A(z)$

$$= -\frac{d}{dw}\frac{1}{1+w} = -\frac{d}{dw}\sum_{n=0}^{\infty}(-1)^n w^n = \sum_{n=1}^{\infty}(-1)^{n-1} n w^{n-1} = 1 - 2A(z) + O(z^2)$$

So

$$(e^z - 1)^{-2} = \frac{1}{z^2}\left(1 - 2(z/2) + O(z^2)\right)$$

(We only need to include the $z/2$ term in $A(z)$, since all other terms in $A(z)$ are of order z^2 or higher, and likewise all terms in $A(z)^m$ for $m \geq 2$.)

So the principal part of $(e^z - 1)^{-2}$ is

$$\frac{1}{z^2}(1 - z) = \frac{1}{z^2} - \frac{1}{z}$$

(3) Find the principal part of $\frac{e^z-1}{e^z+1}$ around $a = i\pi$.

Solution:

$$\frac{e^z-1}{e^z+1} = 1 - \frac{2}{e^z+1}$$

$$e^z = e^{(z-i\pi)}e^{i\pi} = -e^{z-i\pi}$$

Put $w = z - i\pi$.

So

$$1 - e^w = -\sum_{n=1}^{\infty} w^n/n! = -w(1 + A(w))$$

where $A(w)$ is as defined in question 2. Hence

$$\frac{1}{1-e^w} = -\frac{1}{w(1+A(w))}$$

$$= -\frac{1}{w}(1 - A(w) + A(w)^2 - \dots)$$

So the principal part of this expression is the principal part of $-\frac{2}{1-e^w}$ which equals $\frac{2}{w} = \frac{2}{z-i\pi}$.

(4) Locate and classify the singularities of

$$f(z) = \frac{e^{iz}}{(z^2+z+1)^2}$$

Solution:

f is singular when $z^2+z+1 = 0$, in other words when $z = w$ or $z = \bar{w}$, where $w = e^{2\pi i/3}$.

$e^{iz} \neq 0$ when $z = w$ or $z = \bar{w}$. So f has a pole of order 2 at w and a pole of order 2 at $\bar{w}$. couLocate and classify the singularities of

$$f(z) = \frac{z \sin z}{\cos(z) - 1}.$$

Solution:

f has a singularity when $\cos(z) = 1$, or equivalently when $z = 2\pi n$ for n an integer.

$$\cos(z) = \cos(z - 2n\pi)$$

$$\cos(z-2n\pi)-1=\sum_{n=1}^{\infty}(-1)^n\frac{(z-2n\pi)^{2n}}{(2n)!}$$

$=-(z-2n\pi)^2/2+$ higher order terms.

$$\sin(z-2n\pi)=(z-2n\pi)-(z-2n\pi)^3/3!+\dots$$

$\sin(z-2n\pi)$ has a simple zero at $z=2n\pi$. So when $z=0$, f has a removable singularity.

When $z=2n\pi$ $(n\neq 0)$, f has a simple pole.

(5) Locate and classify the singularities of $f(z)=\frac{\cos\pi z}{(z-1)\sin\pi z}$

Solution:

f is singular when $z=1$ or $\sin\pi z=0$. When $z=1$, $\cos\pi z\neq 0$ but $\sin\pi z=0$. Hence $\sin\pi z$ has a zero of order 1 at $z=1$ since $\sin\pi z=-\sin\pi(z-1)$ and $\sin(w)=w-w^3/3!+\dots$ g

where each term in the Taylor expansion of $\sin(w)$ has at least one factor of w.

Hence $\frac{1}{\sin\pi z}$ has a pole of order 1 at $z=1$, and $\frac{\cos\pi z}{(z-1)\sin\pi z}$ has a pole of order 2 at $z=1$.

When n is an integer, $\sin\pi z$ has a zero of order 1 at $z=n$.

When $z=n$, $\cos\pi z\neq 0$. So f has a pole of order 1 at $z=n$ when $n\neq 1$.

(6) Locate and classify the singularities of

$$f(z)=\frac{z\sin z}{\cos(z)-1}$$

Solution:

f has a singularity when $\cos(z)=1$, or equivalently when $z=2\pi n$ for n an integer.

$$\cos(z)=\cos(z-2\pi n)$$

$$\cos(z-2\pi n)-1=\sum_{m=1}^{\infty}(-1)^m\frac{(z-2\pi n)^{2m}}{(2m)!}$$

This equals $-(z-2\pi n)^2/2+$ higher order terms.

$$\sin(z-2\pi n)=(z-2\pi n)-(z-2\pi n)^3/3!+\dots$$

The function $\sin(z-2n\pi)$ has a zero of order 1 at $z=2\pi n$. So when $z=0$, f has a removable singularity.

When $z = 2\pi n$, f has a pole of order 1.

(7) Locate and classify the singularities, including singularities at ∞, of $f(z) = \tan^2(z)$.

Solution:

$$f(z) = \frac{\sin^2(z)}{\cos^2(z)}$$

f is singular when $\cos(z) = 0$ or equivalently $z = \pi/2 + n\pi$.

At these values, $\sin(z) \neq 0$ and $\cos(z)$ has a simple zero since $\cos(z) = \sin(\pi/2 - z)$.

So f has a pole of order 2 at $z = \pi/2 + n\pi$.

Behaviour at $z = \infty$: Set $z = 1/w$.

Define

$$f(z) = \tilde{f}(w) = \frac{\sin^2(1/w)}{\cos^2(1/w)}.$$

In any neighbourhood of $w = 0$ there are infinitely many values of w where $\cos(1/w) = 0$ and $\sin(1/w) \neq 0$. Thus $z = \infty$ is an essential singularity of f.

(8) Locate and classify the singularities (including singularities at ∞) of $f(z) = \cosh^2(1/z)$.

Solution: f is not singular unless $z = 0$ or $z = \infty$.

At $z = 0$ there is an isolated essential singularity because

$$\cosh^2(w) = (e^w + e^{-w})^2 = e^{2w} + e^{-2w} + 2 = \sum_{n=0}^{\infty} 2(\sum_{n=0}^{\infty} w^{2n}/(2n)! + 2$$

so

$$\cosh^2(1/z) = 2\sum_{n=0}^{\infty} z^{-2n}/(2n)! + 2$$

This has infinitely many negative terms, so an isolated essential singularity. at 0.

At $z = \infty$, put $z = 1/w$ wo w is in a neighbourhood of 0.

$$f(z) = \cosh^2(w)$$

so f is smooth at ∞ (and $f(w) \neq 0$ when $w = 0$