University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

MAT C34F 2018/19

Problem Set #5

Due date: Thursday, November 29, 2018 at the beginning of class

(1) Classify the behaviour at ∞ for each of the following functions (zero, pole, removable sing.ularity, essential singu aar.y). If the function has a a zero or pole, give its order):
(i) $\cosh(z)$

Solution:
$\cosh(z) = \cosh(1/w)$ where $w = 1/z$.

$$\cosh(z) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{w^{-2n}}{(2n)!}$$

$w = 0$ (or $z = \infty$) is an isolated essential singularity.

(ii) $\frac{z-1}{z+1}$

Solution:

$$\frac{z-1}{z+1} = \frac{1-w}{1+w}$$

$w = 0$ is not a singular point. It is also not a zero.

(iii) $\frac{z}{z^3+i}$

Solution:

$$\frac{z}{z^3+i} = \frac{1}{w(w^{-3}+i)} = \frac{w^2}{1+iw^2}$$

$w = 0$ is a zero of order w

(iv) $\frac{z^3+i}{z}$

Solution:

$$\frac{z^3+i}{z} = \frac{1+iw^2}{w^2}$$

$w = 0$ is a pole of order 2

(v) $\frac{\sin z}{z^2}$

Solution:

$$\frac{\sin z}{z^2} = w^2 \sin(1/w) = w^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} w^{-2n+1}$$

$w = 0$ is an isolated essential singularity

(2) Find the residues at the poles of the function

$$f(z) = \frac{1}{z^3(z^2+1)}$$

Solution:
At $z = 0$,

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - \dots$$

so the coefficient of z^{-1} in this Laurent series is -1. At $z = +i$,

$$f(z) = \frac{1}{z^2(z+i)(z-i)}$$

so the residue of f at i is $\frac{1}{i^2(i+i)} = -i/2$. At $z = -i$, the residue of f at $-i$ is $\frac{1}{i^2(-i-i)} = +i/2$

(3) Find the residues at the poles of the function

$$\frac{1 - e^{iz}}{z^2}$$

Solution:
The only pole is at $z = 0$. Expanding $e^{iz} = 1 + iz + (iz)^2/2 + \dots$ we find that the principal part of our function is i/z. Hence the residue at 0 is $-i$.

(4) Find the residues at the poles of the function

$$\frac{1}{1 - e^{z^2}}$$

Solution:
We expand $1 - e^{z^2}$ using the Taylor expansion for the exponential function

$$e^{z^2} = 1 + z^2 + z^4/2 + \dots$$

So $1 - e^{z^2} = -z^2(1 + B(z^2))$ where $B(z^2)$ is a power series in z^2 for which every term has a factor z^2.

Now we can invert $1/(1 - e^{z^2}) = (1/z^2)(1/(1 + B(z^2))$

We can deduce from this (and the binomial theorem) that the principal part of $1/(1-e^{z^2})$ is $1/z^2$. So there is a double pole at $z = 0$, and the residue is 0 (because all the powers of z are even).

$z = 0$ is the only singularity of this function (because it is the only value where the denominator is 0).

(5) Compute

$$\int_\gamma \frac{1}{(z-1)^2(z^2+1)} dz$$

where γ is a circle of radius 2 and centre 0, traversed counterclockwise.

Solution: This function has poles at 1 and $\pm i$, all of which are inside this contour. Take $g(z) = \frac{1}{z^2+1}$, and

$$g'(z) = \frac{-2z}{(z^2+1)^2}.$$

Then

$$\text{Res}(f(z); z = 1) = g'(1) = -2/4 = -1/2.$$

The poles at $z = \pm i$ are simple poles so

$$\text{Res}(f(z); i) = \frac{1}{(i-1)^2(i+i)} = \frac{1}{-2i(2i)} = \frac{1}{4}.$$

$$\text{Res}(f(z); z = -i) = \frac{1}{(-i-1)^2(-i-i)} = \frac{1}{(2i)(-2i)} = \frac{1}{4}.$$

So the integral is $2\pi i(-\frac{1}{2} + \frac{1}{2}) = 0$.

(6) Compute

$$\int_\gamma \frac{1}{1+e^z} dz$$

where γ is a circle of radius 8 and center 0 traversed counterclockwise.

Solution:

This function has poles when $e^z = -1 = e^{i\pi}$ in other words $z = i\pi + 2\pi i n$. 2π is approximately 6.28 while $3\pi > 9$. So the only poles inside γ are at $\pm i$.

Residues: $e^z = e^{i\pi} e^{z-i\pi}$ so $1 + e^z = 1 - e^{z-i\pi} = -(z - i\pi)(1+$ higher order).

So the residue at $i\pi$ is -1.

Similarly at $-i\pi$, $e^z = e^{-i\pi} e^{z+i\pi}$ so $1 + e^z = 1 - e^{z+i\pi} = -(z + i\pi)(1+$ higher order). So the residue at $-i\pi$ is -1. So the integral is $2\pi i(-2) = -4\pi i$.

(7) Evaluate the integral

$$\int_0^{2\pi} (\cos^4(\theta) + \sin^4(\theta)) d\theta.$$

by converting it into an integral around a circle of center 0 and radius 1 and applying the residue theorem.

Solution : The integral is

$$\frac{1}{i}\int_\gamma \left[\left(\frac{z+z^{-1}}{2}\right)^4 + \left(\frac{z-z^{-1}}{2i}\right)^4\right]\frac{dz}{z}.$$

This is equal to

$$\frac{1}{16i}\int_\gamma \left[(z^2+2+z^{-2})^2 + (z^2-2+z^{-2})^2\right]$$

$$= \frac{1}{16i}\int_\gamma (2z^4+2z^{-4}+12)\frac{dz}{z}$$

The only term that makes a nonzero contribution is $\int_\gamma \frac{12}{z}dz = 24\pi i$. So the answer is

$$\frac{24\pi i}{16i} = \frac{3\pi}{2}.$$

(8) Prove that

$$\int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{2ab(a+b)}$$

where $a, b > 0$ and $a \neq b$.

Solution:

Let

$$f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}.$$

Then what we want is $\frac{1}{2}\int_\gamma f(z)dz$ where γ is a semicircle in the upper half plane with center 0 and radius $R > a, b$. We need to check that the integral around the semicircular contour γ with radius R tends to 0 as $R \to \infty$. The integral around γ is

$$\frac{1}{2}\int_{z:|z|=R} \frac{1}{(z^2+a^2)(z^2+b^2)}dz.$$

The absolute value of this integral is

$$\leq \frac{1}{(R^2-a^2)(R^2-b^2)}(2\pi R) \leq \frac{1}{R^3}$$

so it vanishes as $R \to \infty$.

To compute the integral around the contour, the poles are at $z = ai$ and $z = ib$, and

$$f(z) = \frac{1}{(z+ai)(z-ai)(z+bi)(z-bi)}$$

The residues are

$$\text{Res}(f(z); z = ai) = \frac{1}{2ai}(ai+bi)(ai-bi)$$

$$= \frac{-1}{2ai(a^2 - b^2)}$$

Similarly

$$\mathrm{Res}(f(z)|z = bi) == \frac{-1}{2bi(b^2 - a^2)}$$

So the sum of residues is

$$\frac{i}{2(a^2 - b^2)} \left[\frac{1}{a} - \frac{1}{b} \right] = \frac{-i}{2ab(a + b)}.$$

So the integral around the contour is

$$2\pi i(\frac{-i}{2ab(a + b)}) = \frac{\pi}{ab(a + b)}.$$

$$\int_0^\infty \frac{dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{2ab(a + b)}.$$

(9) Prove

$$\int_{-\infty}^\infty \frac{\cos(x)}{(x^2 + a^2} dx = \frac{\pi}{a} e^{-a}$$

where $a > 0$.

Solution: Consider

$$\int_\gamma \frac{e^{iz} dz}{(z^2 + a^2)}$$

where γ is a semicircular contour in the upper half plane with center 0 and radius R.

The real part of this is the integral we want, as long as the integral around the semicircular contour tends to 0 as $R \to \infty$. The integral around the semicircle is

$$\int_0^\pi \frac{e^{i(R\cos\theta + iR\sin\theta)} iRe^{i\theta} d\theta}{R^2 e^{2i\theta} + a^2}.$$

Its absolute value is less than or equal to

$$\int_0^\infty \frac{e^{-R\sin\theta} R d\theta}{R^2 - a^2} \leq \frac{\pi R}{R^2 - a^2}.$$

This tends to 0 as $R \to \infty$. This proves the real part of the contour integral is equal to the integral we want.

The residues inside the contour occur at $z = ia$. If

$$f(z) = \frac{e^{iz}}{z^2 + a^2} = \frac{e^{iz}}{(z^+ ia)(z - ia)}$$

then

$$\text{Res}(f(z)|z = ia) = \frac{e^{i(ia)}}{2ai} = \frac{e^{-a}}{2ai}$$

So the contour integral is $\frac{\pi e^{-a}}{a}$.

(10) By integrating $(1+z^n)^{-1}$ around a suitable sector of angle $\frac{2\pi}{n}$, prove that, for $n = 2, 3, \dots,$

$$\int_0^\infty \frac{1}{1+x^n} dx = \frac{\pi}{n \sin(\pi/n)}.$$

Solution: Let $f(z) = \frac{1}{1+z^n}$. The integral equals

$$\int_0^R \frac{dx}{1+x^n} - e^{2\pi i/n} \int_0^R \frac{dx}{1+x^n} + \int_0^{2\pi/n} \frac{Rie^{i\theta} d\theta}{1 + R^n e^{-ni\theta}} d\theta$$

Here the first term is an integral over $z = x \in [0, R]$,

the second is an integral over $z = xe^{2\pi i/n}, x \in [0, R]$, and the third is an integral over $z = Re^{i\theta}, 0 \leq \theta \leq 2\pi/n$.

The third integral tends to 0 as $R \to \infty$ (it is bounded by $R/(R^n - 1)(2\pi/n)$, which tends to 0 as $R \to \infty$, since $n \geq 2$. The integral equals

$$\int_0^{2\pi/n} Rie^{i\theta} d\theta (1 + R^n e^{-in\theta})^{-1}.$$

The third integral tends to 0 as $R \to \infty$, because it is bounded by $\frac{R}{R^n-1} \frac{2\pi}{n}$ and $n \geq 2$. This integral equals

$$(1 - e^{2\pi i/n}) \int_0 -^R (1 + x^n)^{-1} dx$$

The poles occur at z where $z^n = -1 = e^{i\pi}$, in other words where $z = e^{i\pi/n} e^{2\pi i m/n}$ for some integer m. The only pole occurring inside this sector is $m = 0$. The residue is obtained as follows. Let $w = e^{i\pi/n}$.

$$z^n + 1 = (z - w) \prod_{m=1}^{n-1} (z - we^{2\pi i m/n}).$$

So the residue is

$$\frac{1}{\prod_{m=1}^{n-1} (w - we^{2\pi i m/n})} = \frac{1}{aw^{n-1} \prod_{m=1}^{n-1} (1 - e^{2\pi i m/n})}.$$

Thus we see that the residue is

$$\frac{w}{\prod_{m=1}^{n-1} (1 - e^{2\pi i m/n})}$$

Now

$$\prod_{m=1}^{n-1}(1-e^{2\pi im/n}) = \frac{1-x^n}{1-x}|_{x=1}$$

since

$$x^n - 1 = (x-1)\prod_{m=1}^{n-1}(x-e^{2\pi im/n}).$$

But

$$\frac{1-x^n}{1-x} = 1 + x + \cdots + x^{n-1}$$

So

$$\frac{1-x^n}{1-x} = 1 + x + \cdots + x^{n-1}|_{x=1} = n.$$

Hence we find that the residue is $-\frac{w}{n}$. So we have

$$(1-w^2)\int_0^\infty (1+x^n)^{-1}dx = -\frac{2\pi iw}{n}$$

So

$$\int_0^\infty (1+x^n)^{-1}dx = -\frac{2\pi iw}{n(1-w^2)}.$$

$$\frac{2\pi i}{n(w-w^{-1})} = \frac{\pi}{n\sin\pi/n}$$

Likewise

$$\int_\Gamma \frac{z}{1+z^n}dz = \int_0^R \frac{xdx}{1+x^n} - \int_0^R \frac{e^{2\pi i/n}x}{1+x^n}dx + \int_0^\pi (Rie^{i\theta}d\theta)(Re^{i\theta}(1+R^n e^{-in\theta})^{-1}$$

The integral over the circular arc is bounded by

$$\frac{R^2}{R^n-1}$$

This is less than

$$KR^{2-n}$$

for a suitable constant K.

So since $n \geq 3$, this quantity approaches 0 as $R \to \infty$.

The residue at $w = e^{i\pi/n}$ is

$$\frac{e^{i\pi/n}}{\prod_{m=1}^{n-1}(w - we^{2\pi im/n})} = \frac{w}{w^{n-1}\prod_{m=1}^{n-1}(1-e^{2\pi im/n})}.$$

$$= \frac{-w^2}{\prod_{m=1}^{n-1}(1-e^{2\pi im/n})} = -w^2/n.$$

Thus we have

$$(1 - e^{4\pi i/n}) \int_0^\infty \frac{xdx}{1+x^n} = 2\pi i(-\frac{e^{2\pi i/n}}{n})$$

or

$$\int_0^\infty \frac{xdx}{1+x^n} = -\frac{2\pi i}{n(e^{-2\pi i/n} - e^{2\pi i/n})}$$
$$= \frac{\pi}{n\sin(2\pi/n)}.$$