MATC34 Solutions to Assignment 6

1. Find the images of

   (a) $\{z : 0 < \arg(z) < \pi/6\}$

   (b) $D(0; 2)$

   (c) $\{z : 0 < \mathrm{Im}(z) < 1\}$ under $z \mapsto 1/z$.

   Solution: Image under $f : z \mapsto 1/z$ of

   (a) $\{z : 0 < \arg(z) < \pi/6\}$

   $= \{zre^{i\theta} : 0 < \theta < \pi/6\}$ In this case $z^{-1} = r^{-1}e^{-i\theta} : 0 < \theta < \pi/6$ so image is the wedge $\{z : |z| > 0, 0 > \arg(z) > -\pi/6\}$

   (b) $D(0; 2)$

   Soln: $D(0; 2) = \{z = re^{i\theta} | r < 2\}$

   $$1/z = 1/re^{-i\theta} : 1/r > 1/2$$

   so image is $\{z : |z| > 1/2\}$

   (c) $\{z : 0 < \mathrm{Im}(z) < 1\}$

   Soln: $\{z | 0 < \mathrm{Im}(z) < 1\} = \{z = x + iy | 0 < y < 1\}$

   $$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{x - iy}{x^2 + y^2}$$

   The line $z \in \mathbf{R}$ is sent to $\mathbf{R} \setminus \{\mathbf{0}\}$. (The point ∞ is sent to 0.)

   The line $z = x + i$ (where (Im)(z) = 1) contains i, $1 + i$, $-1 + i$, and is sent to the circline containing $1/i$, $1/(1 + i) = (1 - i)/2$ and $1/(-1 + i) = \frac{(-1-i)}{2}$. These three points are

   $$= -i, 1/\sqrt{2}e^{-i\pi/4}, 1/\sqrt{2}e^{-3i\pi/4}.$$

   This is the circle with centre $-i/2$ and radius $1/2$ Under $f(z) = 1/z$, the point $i/2$ is sent to $-2i$. This point is outside the circle described above. So the strip $\{z : 0 < \mathrm{Im}(z) < 1\}$ is sent under $f(z) = 1/z$ to the region below the real axis and outside the circle $\{z : |z + i/2| = 1/2$.

2. Describe the image of $\{z : 0 < \arg(z) < \pi/2\}$ under $z \mapsto w = \frac{z-1}{z+1}$

   Solution: We are looking for the image of $\{z : 0 < \mathrm{Arg}(z) < \pi/2\}$ under $z \mapsto f(z) = \frac{z-1}{z+1}$.

   The first quadrant is bounded by

   (a) the positive real axis

   (b) the positive imaginary axis.

   Image of positive real axis under f:

$$\{\frac{t-1}{t+1} : t \in \mathbf{R}, t > 0\} = 1 - 2/(t+1)\}$$

$$t + 1 \geq 1$$

   implies

$$(t+1)^{-1} \leq 1.$$

   $0 \geq \frac{-2}{t+1} \geq -2$ so $\mathbf{R}^+$ is mapped into the segment $[-1, 1]$.

   Image of positive imaginary axis: $\{\frac{it-1}{it+1} : t \in \mathbf{R}, \mathbf{t} > \mathbf{0}\}$ This is the segment of the circline containing $f(0)$, $f(i)$, $f(\infty)$.

   $f(0) = -1$, $f(i) = \frac{i-1}{i+1} = \frac{\sqrt{2}e^{3i\pi/4}}{\sqrt{2}e^{i\pi/4}}$

$$f(\infty) = 1$$

   A point in the interior of the sector is $z = 1 + i$.

   Then $f(z) = \frac{1+i-1}{1+i-1} = \frac{i}{2+i} = \frac{i(2-i)}{4} = \frac{1/4+i}{2}$. This is in interior of semi-circle. This means that the image of the first quadrant under f is the interior of the intersection of the unit disk with the upper half plane.

3. Describe the image of $\{z : \mathrm{Re}(z) > 0\}$ under $z \mapsto w$ where $\frac{w-1}{w+1} = 2\frac{z-1}{z+1}$

   Solution: We now must solve for w where $\frac{w-1}{w+1} = u$ and $u \in D(0; 2)$.

   $w - 1 = u(w + 1)$

   $w(1 - u) = u + 1$

$$w = \frac{1+u}{1=u}$$

   so we get the image of $D(0; 2)$ under f_3, where $f_3(u) = \frac{1+u}{1-u}$.

This is the region containing $f_3(1) = \infty$ and $f_3(-1) = 0$ and bounded by $f_3(\{z : |z| = 2\})$. This boundary is the circline containing $f_3(2) = (1+2)/(1-2) = -3$, $f_3(-2) = 1-2/(1+2) = -1/3$ and $f_3(2i) = \frac{1+2i}{1-2i} = (1+2i)^2/5$

$$= \frac{-3+4i}{5}$$

$$f_3(-2i) = \frac{1-2i}{1+2i} = \frac{-3-4i}{5}.$$

This circle has centre c, radius r where $r = |-3-c| = |-1/3-c| = |(-3+4i)/5 - c|$. The centre is on the real axis since the real axis is the line of points at the same distance from $(-3+4i)/5$ and its complex conjugage $(-3-4i)/5$. The centre is at the same distance from $-3 = -9/3$and $-1/3$. So $c = -5/3$ The radius is $r = |-3-c| = |-9/3+5/3| = 4/3$

The image is the region outside the circle.

If the centre is c, $c \in \mathbf{R}$ since the real axis is the line of points equidistant from $2i-1$ and $-2i-1$.

So $|2i-1-c| = |(-1+2i)/5 - c|$ So $(1+c)^2 + 4 = (1/5+c)^2 + 4/25$ $1 + 2c - 1/25 - 2/5c = 4(-24)/25$

$$8c/5 = 4(-24/25) - 24/25$$

$$8/5c = -5 \cdot 24/25 = -24/5$$

so $c = 3$. So the radius is $|2i-1+3| = |2i+2| = 2\sqrt{3}$

$$0 \in D(0;2)$$

$f_3(0) = 1$ and $|1-(-3)| = 4 > 2\sqrt{2}$. So the image $f_3(D(0;2))$ is the region outside the circle: $\{z : |z+3| > 2\sqrt{2}\}$.

4. (a) Find Möbius transformations to map
   $1, i, 0$ to $1, i, -1$ respectively
   Soln: Möbius transformation $f(z) = \frac{az+b}{cz+d}$

$$f(1) = 1, f(i) = i, f(0) = -1$$

We have

$$\frac{a+b}{c+d} = 1 \tag{1}$$

or

$$a + b = c + d$$

$$\frac{ai + b}{ci + d} = i \tag{2}$$

or

$$ai + b = -c + id$$

$$b/d = -1$$

or

$$b = -d \tag{3}$$

So substituting (3) in (1) and 2

$$a + b = c - b \tag{4}$$

$$ai + b = -c - ib \tag{5}$$

(4) is equivalent to

$$2b = c - a \tag{6}$$

(5) is equivalent to

$$(1 + i)b = -c - ia \tag{7}$$

$$2b = (-(1 - i)c - i(1 - i)a = (-1 + i)c + (-1 - i)a$$

((7)

$$\times(1 - i)$$

) so

$$c - a = (-1 + i)c + (-1 - i)a \tag{8}$$

so

$$(2 - i)c = -ia$$

so

$$(2i + 1)c = a$$

From (b),

$$2b = c - a = c - (2i + 1)c = -2ic$$

so $b = -i, c = ib$ and $a = (2i + 1)c = (-2 + i)b$ and $d = -b$

So

$$f(z) = \frac{(-2 + i)bz + b}{ibz - b}$$

Set $b = 1$ so

$$f(z) = \frac{(-2+i)z+1}{iz-1}$$

(b) Find Möbius transformations to map $0, 1, \infty$ to $\infty, -i, 1$ respectively Möbius transformation
Soln: $f(z) = \frac{az+b}{cz+d}$ so that $f(0) = \infty$ and $f(1) = -i$ and $f(\infty) = 1$.

$$b/d = \infty \tag{9}$$

so $d = 0$

$$(a+b)/(c+d) = -i \tag{10}$$

so $a + b = -ic$ using equation (9).

$$a/c = 1 \tag{11}$$

so $a = c$ so $c + b = -ic$ so $b = (-1-i)c$

$$f(z) = \frac{cz + (-1-i)c}{cz} = \frac{z + (-1-i)}{z}.$$

Circlines whose images are straight lines:
If $f(z) = w = \frac{z+(-1-i)}{z}$

$$wz = z + (-1-i)$$

$$z(w-1) = -1-i$$

$$z = \frac{-1-i}{w-1}$$

If w is in the line $\{w = a + te^{i\theta} : t \in \mathbf{R}\}$ containing a, $a+b$, $a-b$, then the preimage curve $\{z\}$ is the circline containing

$$\frac{-1-i}{a-1}, \frac{-1-i}{a+b-1}, \frac{-1-i}{a-b-1}$$

5. Find the Möbius transformation mapping $0, 1, \infty$ to $1, 1+i, i$ respectively. Under this mapping what is the image of a circular arc through -1 and $-i$?

Soln:

$$f(z) = \frac{az+b}{cz+d}$$

(a) $f(0) = 1$ iff $b/d = 1$ iff $b = d$

(b) $f(1) = 1 + i$ iff $\frac{a+b}{c+d} = 1 + i$ iff $a + b = (1 + i)c + (1 + i)d$

(c) $f(\infty) = i$ iff $a/c = i$ iff $a = ic$

From the first two equations in the list above,

$$a + b = (1 + i)(-i)a + (1 + i)b$$

$a(1 + i - 1) = ib$ iff $a = b = d$ iff $c = -ia$

So

$$f(z) = \frac{az + a}{-iaz + a} = \frac{z + 1}{-iz + 1}$$

6. Möbius transformations mapping $\{z : \mathrm{Im}(z) > 0\}$ onto $D(0; 1)$ and mapping imaginary axis onto real axis: $f(z) = \frac{az+b}{cz+d}$

   f must map real axis (boundary of $\{z : \mathrm{Im}(z) > 0\}$ onto unit circle $\{z : |z| = 1\}$

   Use text, Example 8.14, Stage 4 (p. 103): $z \mapsto f(z) = \frac{z+i}{z-i}$ maps lower half plane to $D(0; 1)$ So $z \mapsto g(z) = \frac{-z+i}{-z-i}$ maps upper half plane to $D(0; 1)$. It also maps the imaginary axis $i\mathbf{R}$ to the real axis $\mathbf{R}$.

   So our problem reduces to finding the Möbius transformations which map the upper half plane to itself and map $i\mathbf{R}$ to $i\mathbf{R}$. If $f(z) = \frac{az+b}{cz+d}$ maps upper half plane to itself, it must map the real axis to itself:

$$\frac{at + b}{ct + d} = \frac{\bar{a}t + \bar{b}}{\bar{c}t + \bar{d}}$$

$$(at + b)(\bar{c}t + \bar{d}) = (ct + d)(\bar{a}t + \bar{b})$$

$$a\bar{c}t^2 + (b\bar{c} + a\bar{d})t + b\bar{d} = \bar{a}ct^2 + (\bar{b}c + \bar{a}d)t + \bar{b}d$$

7. (i) Find the image of $\{z : 0 < \mathrm{Arg}(z) < \pi/4\}$ under $z \mapsto iz^4$

   Solution: If $f(z) = z^4$, then the sector $\{z : 0 < \mathrm{Arg}(z) < \pi/4\}$ maps to the upper half plane $(z : Im(z) > 0)$. Multiplying it by i maps to the left half plane (union of the second and third quadrants) $\{z : Re(z) < 0\}$

   (ii) Find the image of $\{z : 0 < \mathrm{Re}(z) < 1, 0 < \mathrm{Im}(z) < \pi/2\}$ under $z \mapsto e^z$

Solution: We showed in class that $a < \text{Re} < b$ maps under $f(z) = e^z$ to the ring $e^a < |z| < e^b$. We also showed that $c < \text{Im}(z) < d$ maps under $f(z) = e^z$ to $c < \text{Arg}(z) < d$. So the image is the intersection of the ring $1 < |z| < e$ with the first quadrant $0 < \text{Arg}(z) < \pi/2$.

8. Construct a conformal map onto $D(0;1)$ for $\{z : -1 < \text{Re}(z) < 1\}$

   Solution: The map $f(z) = z + i$ sends the strip $x + iy : -1 < y < 1$ to $x + iy : 0 < y < 2$. The map $g(z) = (\pi/2)z$ sends $0 < y < 2$ to $0 < y < \pi$. The map $h(z) = e^z$ sends $0 < y < \pi$ to the upper half plane. The map $j(z) = \frac{z-i}{z+i}$ sends the upper half plane to the unit disk (as discussed in class).

   So the map we want is the composition $j \circ h \circ g \circ f$.

9. Check that each of the following functions is harmonic on the indicated set, and find a holomorphic function of which it is the real part.

   (i) $\sin(x^2 - y^2)e^{-2xy}$

   Soln:

   Use the fact that

   if $Z = X + iY$ for real variables X and Y,

$$\exp(Z) = e^X(\cos Y + i \sin Y)$$

   Also

$$z^2 = (x^2 - y^2) + 2ixy = X + iY$$

   where $X = x^2 - y^2$ and $Y = 2xy$.. So our function is $Re(-i\exp(Z)) = u$ where $f = -i\exp(z^2)$. Then the harmonic conjugate is

$$v = \text{Im}(f) = \text{Im}(-i\exp(z^2) = \text{Im}(-ie^{x^2-y^2}\cos(2xy)).$$

   (ii) $\log(x^2 + y^2)^{3/2}$ (on the open first quadrant).

   Soln:

   If $u = (\ln(x^2 + y^2))^a$ for some real number a (here $a = 3/2$) then

$$u_x = a\frac{\partial}{\partial x}\ln(x^2 + y^2) = \frac{a}{x^2 + y^2}(2x)$$

so

$$u_{xx} = \frac{2a}{x^2+y^2} - \frac{2ax}{(x^2+y^2)^2}(2x)$$

$$= \frac{2a}{x^2+y^2}\left(1 - \frac{2x^2}{x^2+y^2}\right)$$

$$u_{xx} + u_{yy} = \frac{2a}{x^2+y^2}\left(2 - 2\frac{(x^2+y^2)}{x^2+y^2}\right) = 0$$

So u is harmonic.

Construct its harmonic conjugate v:

$u_x = v_y$

We saw above that

$$u_x = \frac{2xa}{x^2(1+z^2)}$$

where $z = y/x$

So

$$v = x\int \frac{2xa}{x^2(1+z^2)} dz = \frac{2x^2a}{x^2}\arctan(y/x) + h(x)$$

Hence

$$v_x = h'(x) + 2a(\frac{-y}{x^2+y^2}$$

This must be equal to $-u_y$ which is

$$\frac{-2ay}{x^2+y^2}$$

Hence $h'(x) = 0$ and (

$$v(x,y) = 2a\arctan(y/x) + \text{constant}$$

gives $u + iv$ is a holomorphic function of z.