University of Toronto Scarborough

CSC B36 **Final Exam** **19 December 2023**

Student Number: : : : : : : : : : : :

Last (Family) Name: ______________________

First (Given) Name: ______________________

Do not turn this page until you are told to do so.

In the meantime, complete the above and read the rest of this cover page.

Aids allowed: None.

Duration: 180 minutes.

There are 9 pages in this exam. Each is numbered at the bottom. *When you receive the signal to start, please check that you have all the pages.*

You will be graded on your mastery of course material as taught in class. So you need to demonstrate this. Unless otherwise stated, you must explain or justify every answer.

Answer each question in the space provided. The last page is intentionally left blank in case you need more space for one of your answers. *You must clearly indicate where your answer is and what part should be marked.* **What you write on backs of pages will not be graded.**

Question	Your mark	Out of
1.		8
2.		12
3.		9
4.		6
5.		7
6.		8
7.		10
Total		60

1. **[8 marks]** Consider the following recurrence defining a function $f : \mathbb{N} \to \mathbb{N}$.

$$f(n) = \begin{cases} 5, & \text{if } n = 0; \\ 1 + \sum_{i=0}^{n-1} f(i), & \text{if } n > 0. \end{cases}$$

Use induction to prove that for all *positive* integers n, $f(n) = 3 \cdot 2^n$.

You may use the following fact about geometric sums without proof.

Geometric Sum Fact: For all positive integers m, $\sum_{i=1}^{m} 2^i = 2^{m+1} - 2$.

2. **[12 marks]** Use methods from class to prove that the program below is correct with respect to its given specification.

   ▷ Precondition: a and b are positive integers.

   ▷ Postcondition: Return a pair of integers (p, q) such that $pa + qb = \gcd(a, b)$.

```
DECOMP(a, b)
    x = a;    y = b;    p = 1;    q = 0;    r = 0;    s = 1
    while x ≠ y:
        if x < y:
            y = y − x;    r = r − p;    s = s − q
        else:
            x = x − y;    p = p − r;    q = q − s
    return (p, q)
```

   You may use the following fact (from a tutorial exercise) without proof.

   Fact from class: If a and b are distinct nonzero integers, then $\gcd(a, b) = \gcd(a, b - a)$.

[more space available on next page . . .]

[... additional space for question 2]

3. **[9 marks]** Let $\Sigma = \{0, 1\}$. Consider the following language operations.

$f_1(L) = \{x : \text{for some } v, w \in \Sigma^*,\ vw \in L \text{ and } x = v0w\}$.

$f_2(L) = \{x : \text{for some } v, w, y \in \Sigma^*,\ vwy \in L \text{ and } x = v0w0y\}$.

Nick wanted to prove that the following predicate holds for all regular expressions over Σ.

$P(R)$: There exist regular expressions R_1 and R_2 such that
$\mathcal{L}(R_1) = f_1(\mathcal{L}(R))$ and $\mathcal{L}(R_2) = f_2(\mathcal{L}(R))$.

He wrote some rough notes for his proof, but got distracted before long. Here is what he wrote.

BASE CASES (4 of them):

For $R = \emptyset$, let $R_1 =$ ____________________ and $R_2 =$ ____________________.

For $R = \epsilon$, let $R_1 =$ ____________________ and $R_2 =$ ____________________.

For $R = 0$, let $R_1 =$ ____________________ and $R_2 =$ ____________________.

For $R = 1$, let $R_1 =$ ____________________ and $R_2 =$ ____________________.

INDUCTION STEP:

Let S and T be regular expressions.
Suppose $P(S)$ and $P(T)$. I.e., there exist regular expressions S_1 and S_2 and T_1 and T_2 such that
$\mathcal{L}(S_1) = f_1(\mathcal{L}(S))$ and $\mathcal{L}(S_2) = f_2(\mathcal{L}(S))$ and $\mathcal{L}(T_1) = f_1(\mathcal{L}(T))$ and $\mathcal{L}(T_2) = f_2(\mathcal{L}(T))$.
WTP: $P(R)$ holds for each of the 3 cases (i) $R = S + T$, (ii) $R = ST$ and (iii) $R = S^*$.

For $R = S+T$, let $R_1 =$ ____________________ and $R_2 =$ ____________________.

For $R = ST$, let $R_1 =$ ____

and $R_2 =$ ____.

For $R = S^*$, let $R_1 =$ ____

and $R_2 =$ ____.

Please complete Nick's notes by filling in the blanks. No justification is required.

4. **[6 marks]** Let $\Sigma = \{0, 1\}$. Let $L_4 = \{x \in \Sigma^* : \text{both } 00 \text{ and } 11 \text{ are substrings of } x\}$.
Using as few states as possible, give a DFSA M such that $\mathcal{L}(M) = L_4$.
Justify the correctness of your DFSA with a state invariant.

5. **[7 marks]** Let $\Sigma = \{7, 9\}$. For arbitrary strings $x, y \in \Sigma^*$, we define $\#_y(x)$ to be $|\{(u, v) : x = uyv\}|$.
Let $L_5 = \{x \in \Sigma^* : \#_{77}(x) = \#_9(x) + 1\}$. Let $L_0 = \{x \in \Sigma^* : \#_{77}(x) = \#_9(x)\}$.

   Using as few productions as possible, give a CFG that generates L_5 according to the following design.

   - S generates L_5.
   - A generates L_0.
   - B generates $\{x \in L_0 : x \text{ starts with } 7\}$.
   - C generates $\{x \in L_0 : x \text{ ends with } 7\}$.

   Use the left-to-right method for every variable except C. Use the right-to-left method for C.
No justification is required.

6. **[8 marks]** Let $\Sigma = \{7, 9\}$. For arbitrary strings $x, y \in \Sigma^*$, we define $\#_y(x)$ to be $|\{(u, v) : x = uyv\}|$.
Let $L_6 = \{x \in \Sigma^* : \#_{77}(x) = \#_9(x) + 1\}$. *This is the same language as L_5 from question 5.*

   Using as few states as possible, give a (nondeterministic) PDA that accepts L_6.
No justification is required, but it may help you earn part marks if your PDA is incorrect.

7. **[10 marks total]** Reminder: *Use good proof structure and counterexamples as appropriate.*

Consider a first-order language with a binary predicate B and the equality predicate $=$.
We define formulas F_1 and F_2 as follows.

F_1: $(\ B(x, y) \rightarrow \neg B(y, x) \ )$ $\qquad$ F_2: $(\ \neg = (x, y) \rightarrow (B(x, y) \leftrightarrow \neg B(y, x)) \ )$

(a) **[5 marks]** Does F_1 logically imply F_2? Justify your answer.

(b) **[5 marks]** Does $\forall x \ \forall y \ F_1$ logically imply $\forall x \ \neg B(x, x)$? Justify your answer.

This page is intentionally left blank in case you need more space for one of your answers.