University of Toronto Scarborough

CSC B36 **Final Exam** **21 August 2024**

Student Number: : : : : : : : : : : :

Last (Family) Name: ______________________________

First (Given) Name: ______________________________

Do not turn this page until you are told to do so.
In the meantime, complete the above and read the rest of this cover page.

Aids allowed: None.

Duration: 180 minutes.

There are 9 pages in this exam. Each is numbered at the bottom. *When you receive the signal to start, please check that you have all the pages.*

You will be graded on your mastery of course material as taught in class. So you need to demonstrate this. Unless otherwise stated, you must explain or justify every answer.

Answer each question in the space provided. The last page is intentionally left blank in case you need more space for one of your answers. *You must clearly indicate where your answer is and what part should be marked.* **What you write on backs of pages will not be graded.**

Question	Your mark	Out of
1.		8
2.		12
3.		8
4.		7
5.		8
6.		7
7.		10
Total		60

1. **[8 marks]** Here is a structural induction definition of a set of ordered pairs of natural numbers.

   Let S be the smallest set such that:

   BASIS: $(0, 0) \in S$.

   INDUCTION STEP: If $(v, w) \in S$, then (A) $(v + 1, w) \in S$ and (B) $(0, v + 1) \in S$.

   For $n \in \mathbb{N}$, we define T_n to be the set $\{(x, y) \in \mathbb{N}^2 : x + y = n\}$.

   Use induction to prove that for all $n \in \mathbb{N}$, $T_n \subseteq S$. Use good proof structure as shown in class.

2. **[12 marks]** Use methods from class to prove that the program below is correct with respect to its given specification. *You may omit step 2 (the induction proof).*

$\triangleright$ Precondition: L is a list of integers, $n = \text{len}(L) > 0$, $x \in \mathbb{Z}$.
$\triangleright$ Postcondition: Return $|\{j : 0 \leq j < n \text{ and } L[j] = x\}|$.
$\triangleright$ I.e., return the number of occurrences of x in L.

```
NumX(L, n, x)
    c = 0;    i = 0;    b = 0
    while i · b < n:
        if L[i] == x:    c = c + 1
        i = i + 2
        if i ≥ n and b == 0:
            i = 1;    b = 1
    return c
```

Advice: Relate the value of b to the parity of i.

[more space available on next page . . .]

[... additional space for question 2]

3. **[8 marks total; 4 for each part]** For each of these implications, state whether it holds for arbitrary regular expressions R, S, and justify your claim.

   (a) If $RS \equiv RSS$, then $RS \equiv RSSS$.

   (b) If $RS \equiv RSS$, then $RS \equiv RS^*$.

4. **[7 marks]** Let $\Sigma = \{0, 1\}$. Given a language L over Σ, we define $f(L)$ to be:

$$\{1y1 : 0y0 \in L\} \cup \{0y0 : 1y1 \in L\}$$

Let $M = (Q, \Sigma, \delta, s, F)$ be a DFSA.

Describe clearly how to construct an NFSA $M' = (Q', \Sigma, \delta', s', F')$ such that $\mathcal{L}(M') = f(\mathcal{L}(M))$.

No justification is required, but part marks may be given for it if your construction is incorrect.

5. **[8 marks]** Let $\Sigma = \{3, 7\}$. For arbitrary strings $x, y \in \Sigma^*$, we define $\#_y(x)$ to be $|\{(u, v) : x = uyv\}|$.

Let $L_5 = \{x \in \Sigma^* : 2\,\#_{77}(x) = \#_{73}(x) + 1\}$. Let $L_0 = \{x \in \Sigma^* : 2\,\#_{77}(x) = \#_{73}(x)\}$.

Nick used the design below and the left-to-right method to create a CFG that generates L_5.

S generates L_5. For $i, j \in \Sigma$, A_{ij} generates $\{x \in L_0 : x \text{ starts with } i,\ x \text{ ends with } j\}$.

Before anyone saw his CFG, the **Grammar Mangler** struck! Destroyed beyond recognition were all the S productions and all but one production for each of variables A_{33}, A_{73}, A_{77}. The A_{37} productions were untouched. Here is what remains of Nick's CFG.

$S \to \cdots$

$A_{33} \to 3A_{73}, \cdots$

$A_{37} \to 3A_{77},\ 3A_{37}$

$A_{73} \to 7A_{77}A_{37}A_{33}, \cdots$

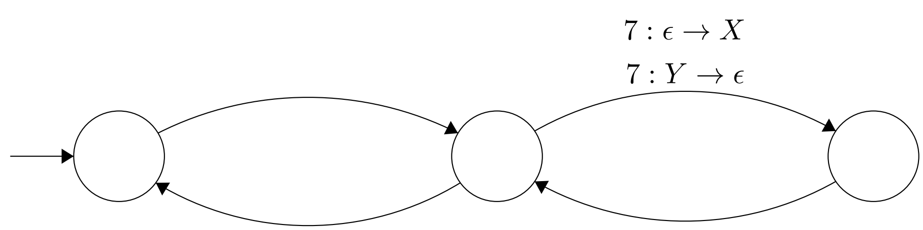
$A_{77} \to 7A_{77}A_{37}A_{37}, \cdots$

Please restore Nick's CFG where you see "$\cdots$". No justification is required.

6. **[7 marks]** Let $\Sigma = \{3, 7\}$. For arbitrary strings $x, y \in \Sigma^*$, we define $\#_y(x)$ to be $|\{(u, v) : x = uyv\}|$.

Let $L_6 = \{x \in \Sigma^* : 2\,\#_{77}(x) = \#_{73}(x) + 1\}$. *This is the same language as L_5 from question 5.*

Nick created a PDA that accepts L_6. Before anyone saw it, the **Automaton Mangler** struck! Destroyed beyond recognition were an accepting state, 3 transition arrows, and all but two of the transition labels. Here is what remains of Nick's PDA.



Please restore Nick's PDA. No justification is required.

7. **[10 marks total]** Recall the unary connectives $\underline{0}$ and $\underline{1}$ from class. We introduce a new binary connective $\ll$ (called *less*). The truth tables defining all three are given here.

P	$\underline{0}P$	$\underline{1}P$
0	0	1
1	0	1

P	Q	$P \ll Q$
0	0	0
0	1	1
1	0	0
1	1	0

For each of the following sets of connectives, state whether it is complete or not complete, and justify your claim with an informal proof as shown in class.

(a) **[5 marks]** $\{\underline{0}, \ll\}$.

(b) **[5 marks]** $\{\underline{1}, \ll\}$.

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