**University of Toronto Scarborough**

**CSC B36**                  **Final Exam**                  **19 August 2025**

**Student Number:** : : : : : : : : : :

**Last (Family) Name:** _____

**First (Given) Name:** _____

---

***Do not turn this page until you are told to do so.***
In the meantime, complete the above and read the rest of this cover page.

---

**Aids allowed:** None. Anything other than your pen/pencil and your TCard is unauthorized.

**Duration:** 180 minutes.

There are 8 pages in this exam. Each is numbered at the bottom. *When you receive the signal to start, please check that you have all the pages.*

You will be graded on your mastery of course material as taught in class. So you need to demonstrate this. Unless otherwise stated, you must explain or justify every answer.

Answer each question in the space provided. The last page is intentionally left blank in case you need more space for your answers. *You must clearly indicate where your answers are and what should be marked.* **What you write on backs of pages will <u>not</u> be graded.**

<u>Definition from class</u>: Given an alphabet $\Sigma$ and arbitrary strings $x, y \in \Sigma^*$, we define $\#_y(x)$ to be
$$|\{(u,v) : \ x = uyv\}|.$$

| Question | Your mark | Out of |
|---|---|---|
| 1. | | 8 |
| 2. | | 12 |
| 3. | | 8 |
| 4. | | 7 |
| 5. | | 8 |
| 6. | | 7 |
| 7. | | 10 |
| Total | | 60 |

1. **[8 marks]** We define a subset of $\mathbb{N}^2$ as follows.

    Let $V$ be the smallest set such that:
        BASIS: $(0,0)$, $(1,1)$, $(2,2) \in V$.
        INDUCTION STEP: If $(s,t) \in V$, then $(s+1, t+2), (s+2, t+1) \in V$.

    Use induction to prove that for every $(x,y) \in V$, $x \leq 2y$ and $y \leq 2x$.

2. **[12 marks]** Let $\Sigma_3 = \{0, 1, 2\}$. For a nonempty string $x \in \Sigma_3^*$ (we call $x$ a *ternary* string), we define $W(x)$ to be the number represented in base 3 by $x$. More formally, if $x = x[0]\, x[1]\, \cdots\, x[n-1]$, where $n > 0$ and each $x[i] \in \Sigma_3$, then $W(x) = \sum_{i=0}^{n-1} \left( x[n-1-i] \cdot 3^i \right)$.

Use methods from class to prove that the program below is correct with respect to its specification.

    $\triangleright$ Precondition: $x$ is a nonempty ternary string.
    $\triangleright$ Postcondition: Return TRUE if $W(x)$ is odd; Return FALSE if $W(x)$ is even.
    IsOdd($x$)

```
1      i = 1;    r = (x[0] == 1)     ▷ r is a boolean variable
2      while i < len(x):
3          r = r ⊕ (x[i] == 1)     ▷ ⊕ means exclusive or
4          i = i + 1
5      return r
```

*[. . . additional space for question 2]*

3. [**8 marks**] Let $\Sigma = \{0,1\}$. Given a language $L$ over $\Sigma$, we define $f(L)$ to be
$$\{x : \text{there exist } y \in \Sigma^* \text{ and } z \in \mathcal{L}(1^*) \text{ such that } y0z \in L \text{ and } x = yz\}.$$

Informally, to get a string in $f(L)$, we take a string in $L$ with one or more 0s and delete its last 0.

Let $M = (Q, \Sigma, \delta, s, F)$ be a DFSA.

Describe clearly how to construct an NFSA $M' = (Q', \Sigma, \delta', s', F')$ such that $\mathcal{L}(M') = f(\mathcal{L}(M))$.

No justification is required, but part marks may be given for it if your construction is incorrect.

4. **[7 marks]** Let $\Sigma = \{0, 1\}$. Let $L_4 = \{x \in \Sigma^* : \#_{011}(x) = 0 < \#_{100}(x)\}$.

Give a regular expression that denotes $L_4$ and briefly explain why it is correct.
For full credit, your regex must not be longer than 26 symbols (implied parentheses do not count).
A bonus mark will be awarded if your regex is both correct and shorter than 24 symbols.

5. **[8 marks]** Let $\Sigma = \{0, 1\}$. Let $L_5 = \{x \in \Sigma^* : \#_{00}(x) = \#_{110}(x)\}$.

Here is Nick's design, with 7 variables, for a CFG that generates $L_5$.

$S$ generates $L_5$.
$A_{00}$ generates $\{x \in L_5 : x$ starts with 0, $x$ ends with 0$\}$.
$A_{01}$ generates $\{x \in L_5 : x$ starts with 0, $x$ ends with 1$\}$.
$A_{10}$ generates $\{x \in L_5 : x$ starts with 1 but not 10, $x$ ends with 0$\}$.
$A_{11}$ generates $\{x \in L_5 : x$ starts with 1 but not 10, $x$ ends with 1$\}$.
$A_{10:0}$ generates $\{x \in L_5 : x$ starts with 10, $x$ ends with 0$\}$.
$A_{10:1}$ generates $\{x \in L_5 : x$ starts with 10, $x$ ends with 1$\}$.

Please help Nick get started on his CFG by using the left-to-right (L2R) method to "code" the productions for the variables listed below.

$S \rightarrow$

$A_{00} \rightarrow$

$A_{10} \rightarrow$

$A_{10:1} \rightarrow$

No justification is required.

6. [**7 marks**] Let $\Sigma = \{0, 1\}$. Let $L_6 = \{x \in \Sigma^* : \#_0(x) = 2\#_{10}(x) + 1\}$.
   For your information, if $x = 10$, then $\#_0(x) = \#_{10}(x) = 1$. Also, $1000, 0100, 0010 \in L_6$.

   Nick created a DPDA that accepts $L_6$. Before anyone saw it, the **Automaton Mangler** struck! Destroyed beyond recognition were the initial state plus one transition from it, and all except 4 transition labels. Here is what remains of Nick's DPDA.
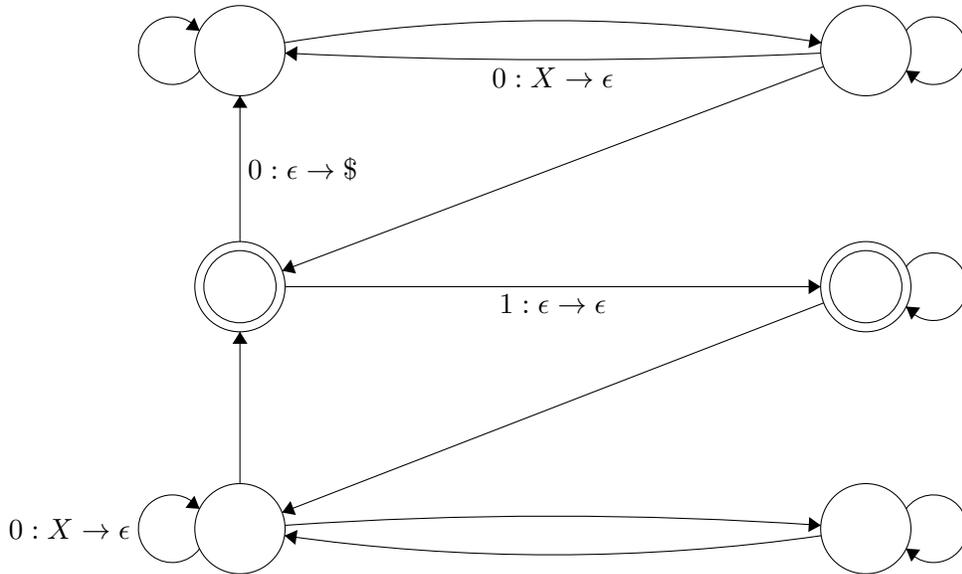


   Please restore Nick's DPDA. No justification is required.

7. [**10 marks total; 5 for each part**]

   (a) <u>Informally</u> prove that $\{\rightarrow, \oplus\}$ is a complete set of connectives. $\oplus$ is the *exclusive or* connective.

   (b) <u>Informally</u> prove that $\{\wedge, \vee, \leftrightarrow\}$ is **not** a complete set of connectives.

*This page is intentionally left blank in case you need more space for one of your answers.*